Regular Expressions

Recap from Last Time

Regular Languages

- A language L is a ***regular language*** if there is a DFA D such that $\mathcal{L}(D) = L$.
- ***Theorem:*** The following are equivalent:
  - L is a regular language.
  - There is a DFA for L.
  - There is an NFA for L.

Language Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, then wx is the ***concatenation*** of w and x.
- If L_1 and L_2 are languages over Σ, the ***concatenation*** of L_1 and L_2 is the language L_1L_2 defined as

$$L_1L_2 = \{ wx \mid w \in L_1 \text{ and } x \in L_2 \}$$

- Example: if L_1 = { **a**, **ba**, **bb** } and L_2 = { **aa**, **bb** }, then

L_1L_2 = { **aaa**, **abb**, **baaa**, **babb**, **bbaa**, **bbbb** }

Lots and Lots of Concatenation

- Consider the language L = { **aa**, **b** }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ **aaaa**, **aab**, **baa**, **bb** }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**, **aabaab**, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**, **baabb**, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

Language Exponentiation

- We can define what it means to "exponentiate" a language as follows:
- $L^0 = \{\varepsilon\}$
  - The set containing just the empty string.
  - Idea: Any string formed by concatenating zero strings together is the empty string.
- $L^{n+1} = LL^n$
  - Idea: Concatenating (n+1) strings together works by concatenating n strings, then concatenating one more.
- ***Question:*** Why define $L^0 = \{\varepsilon\}$?
- ***Question:*** What is $\emptyset^0$?

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

- Mathematically:

$$w \in L^* \quad \text{iff} \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, all possible ways of concatenating zero or more strings in L together, possibly with repetition.
- ***Question:*** What is $\emptyset^0$?

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

…

}

Think of L* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - L_1L_2
  - L_1^*
- These properties are called ***closure properties of the regular languages***.

New Stuff!

Another View of Regular Languages

Rethinking Regular Languages

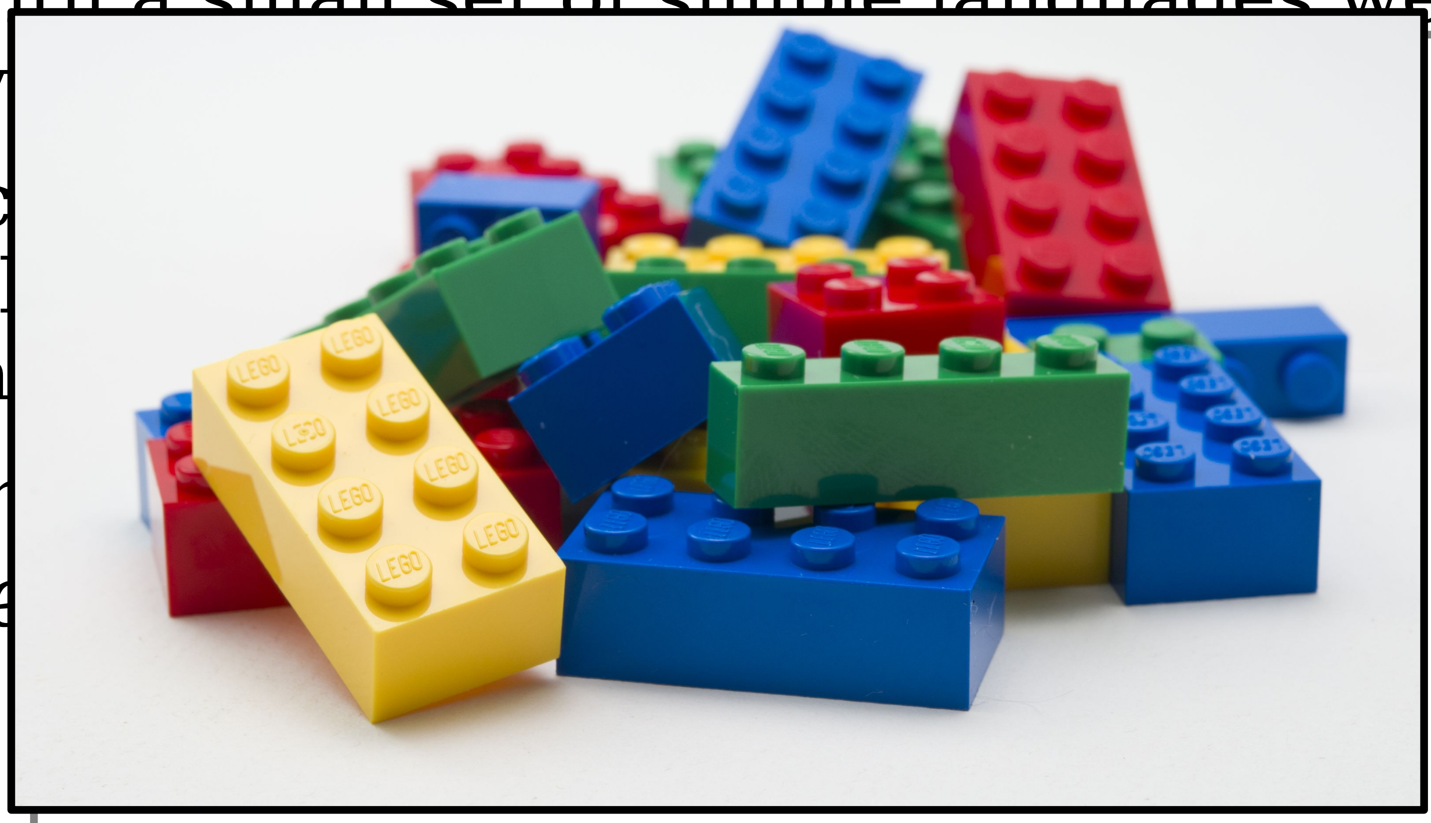
- We currently have several tools for showing a language L is regular:
  - Construct a DFA for L.
  - Construct an NFA for L.
  - Combine several simpler regular languages together via closure properties to form L.
- We have not spoken much of this last idea.

Constructing Regular Languages

- ***Idea:*** Build up all regular languages as follows:
  - Start with a small set of simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more elaborate languages.
- *A bottom-up approach to the regular languages.*

Constructing Regular Languages

- ***Idea:*** Build up all regular languages as follows:
  - Start with a small set of simple languages we already
  - Using c simple l elabora
- *A bottom language*



Regular Expressions

- ***Regular expressions*** are a way of describing a language via a string representation.
- They're used extensively in software systems for string processing and as the basis for tools like `grep` and `flex`.
- Conceptually, regular expressions are strings describing how to assemble a larger language out of smaller pieces.

Atomic Regular Expressions

- The regular expressions begin with three simple building blocks.
- The symbol **Ø** is a regular expression that represents the empty language Ø.
- For any **a** ∈ Σ, the symbol **a** is a regular expression for the language {**a**}.
- The symbol **ε** is a regular expression that represents the language {ε}.
  - ***Remember:* {ε} ≠ Ø!**
  - ***Remember:* {ε} ≠ ε!**

Compound Regular Expressions

- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1R_2}$ is a regular expression for the *concatenation* of the languages of R_1 and R_2.
- If R_1 and R_2 are regular expressions, $\boldsymbol{R_1 \cup R_2}$ is a regular expression for the *union* of the languages of R_1 and R_2.
- If R is a regular expression, $\boldsymbol{R^*}$ is a regular expression for the *Kleene closure* of the language of R.
- If R is a regular expression, $\boldsymbol{(R)}$ is a regular expression with the same meaning as R.

Regular Expression Examples

- The regular expression **hello∪goodbye** represents the regular language { **hello**, **goodbye** }.
- The regular expression **helloo*** represents the regular language { **hello**, **helloo**, **hellooo**, … }.
- The regular expression **(bye)*** represents the regular language { **ε**, **bye**, **byebye**, **byebyebye**, … }.

Operator Precedence

- Regular expression operator precedence:

$$(R)$$

$$R^*$$

$$R_1R_2$$

$$R_1 \cup R_2$$

- So **ab*c∪d** is parsed as **((a(b*))c)∪d**

Regular Expressions, Formally

- The ***language of a regular expression*** is the language described by that regular expression.
- Formally:
  - $\mathcal{L}(\boldsymbol{\varepsilon}) = \{\varepsilon\}$
  - $\mathcal{L}(\boldsymbol{\emptyset}) = \emptyset$
  - $\mathcal{L}(\mathbf{a}) = \{\mathbf{a}\}$
  - $\mathcal{L}(R_1R_2) = \mathcal{L}(R_1)\ \mathcal{L}(R_2)$
  - $\mathcal{L}(R_1 \cup R_2) = \mathcal{L}(R_1) \cup \mathcal{L}(R_2)$
  - $\mathcal{L}(R^*) = \mathcal{L}(R)^*$
  - $\mathcal{L}((R)) = \mathcal{L}(R)$

> Worthwhile activity: Apply this recursive definition to
>
> **a(b∪c)((d))**
>
> and see what you get.

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

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(a ∪ b)*aa(a ∪ b)*

Designing Regular Expressions

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Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

(a ∪ b)*aa(a ∪ b)*

bbabbbaabab
aaaa
bbbbbabbbbaabbbb

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { w ∈ Σ* | w contains **aa** as a substring }.

(a ∪ b)*aa(a ∪ b)*

bbabbbaabab
aaaa
bbbbbabbbbaabbbbb

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{\, w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

Σ*aaΣ*

bbabbbaabab
aaaa
bbbbbabbbbaabbbbb

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{\ w \in \Sigma^* \mid |w| = 4\ \}$.

Designing Regular Expressions

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

Let $L = \{\ w \in \Sigma^* \mid |w| = 4\ \}$.

The length of a string w is denoted $|w|$

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Designing Regular Expressions

- Let $\Sigma = \{\texttt{a}, \texttt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$$\mathbf{\Sigma\Sigma\Sigma\Sigma}$$

Designing Regular Expressions

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$$\Sigma\Sigma\Sigma\Sigma$$

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$$\Sigma\Sigma\Sigma\Sigma$$

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

ΣΣΣΣ

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Σ^4

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

Σ^4

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { w ∈ Σ* | w contains at most one **a** }.

Here are some candidate regular expressions for the language L. Which of these are correct?

Σ*aΣ*
b*ab* ∪ b*
b*(a ∪ ε)b*
b*a*b* ∪ b*
b*(a* ∪ ε)b*

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

b*(a ∪ ε)b*

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
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b*(a ∪ ε)b*

Designing Regular Expressions

- Let Σ = {**a**, **b**}.
- Let L = { w ∈ Σ* | w contains at most one **a** }.

b*(a ∪ ε)b*

bbbbabbb
bbbbbb
abbb
a

Designing Regular Expressions

- Let Σ = {a, b}.
- Let L = { w ∈ Σ* | w contains at most one a }.

b*(a ∪ ε)b*

bbbbabbb
bbbbbb
abbb
a

Designing Regular Expressions

- Let Σ = {a, b}.
- Let L = { w ∈ Σ* | w contains at most one a }.

b*a?b*

bbbbabbb
bbbbbb
abbb
a

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents “some letter.”
- Let's make a regex for email addresses.

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cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let $\Sigma = \{$ **`a`**, **`.`**, **`@`** $\}$, where **`a`** represents "some letter."
- Let's make a regex for email addresses.

`cs103``@cs.stanford.edu`

`first``.middle.last@mail.site.org`

`dot``.at@dot.com`

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
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aa*

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aa*

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dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
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```
aa* (.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
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```
aa* (.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa* (.aa*)* @
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { `a`, `.`, `@` }, where `a` represents "some letter."
- Let's make a regex for email addresses.

```
aa* (.aa*)* @
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa* (.aa*)* @ aa*.aa*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa* (.aa*)* @ aa*.aa*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa*(.aa*)*@aa*.aa*(.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { `a`, `.`, `@` }, where `a` represents "some letter."
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```
aa* (.aa*)* @ aa*.aa* (.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

a⁺ (.aa*)* @ aa*.aa* (.aa*)*

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
a+ (.aa*)* @ aa*.aa* (.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$\mathtt{a}^+ \;\; (\mathtt{.a}^+)^* \;\; \mathtt{@} \;\; \mathtt{a}^+ \, \mathtt{.a}^+ \;\; (\mathtt{.a}^+)^*$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$a^+ \ (.a^+)^* \ @ \ a^+ \ .a^+ \ (.a^+)^*$$

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A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$\mathtt{a^+\ (.a^+)^*\ @\ a^+\ .a^+\ (.a^+)^*}$$

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$$\mathtt{a}^+ \ (\mathtt{.a}^+)^* \ \mathtt{@} \ \ \mathtt{a}^+ \ \boxed{\mathtt{.a}^+ \ (\mathtt{.a}^+)^*}$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
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A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$\texttt{a}^+ \ (\texttt{.a}^+)^* \ \texttt{@} \ \texttt{a}^+ \ (\texttt{.a}^+)^+$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let Σ = { a, ., @ }, where a represents "some letter."
- Let's make a regex for email addresses.

a⁺ (.a⁺)* @ a⁺ (.a⁺)⁺

cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com

A More Elaborate Design

- Let $\Sigma = \{$ a, ., @ $\}$, where a represents "some letter."
- Let's make a regex for email addresses.

$$a^+ (.a^+)^* @ a^+ (.a^+)^+$$

cs103@cs.stanford.edu
first.middle.last@mail.site.org
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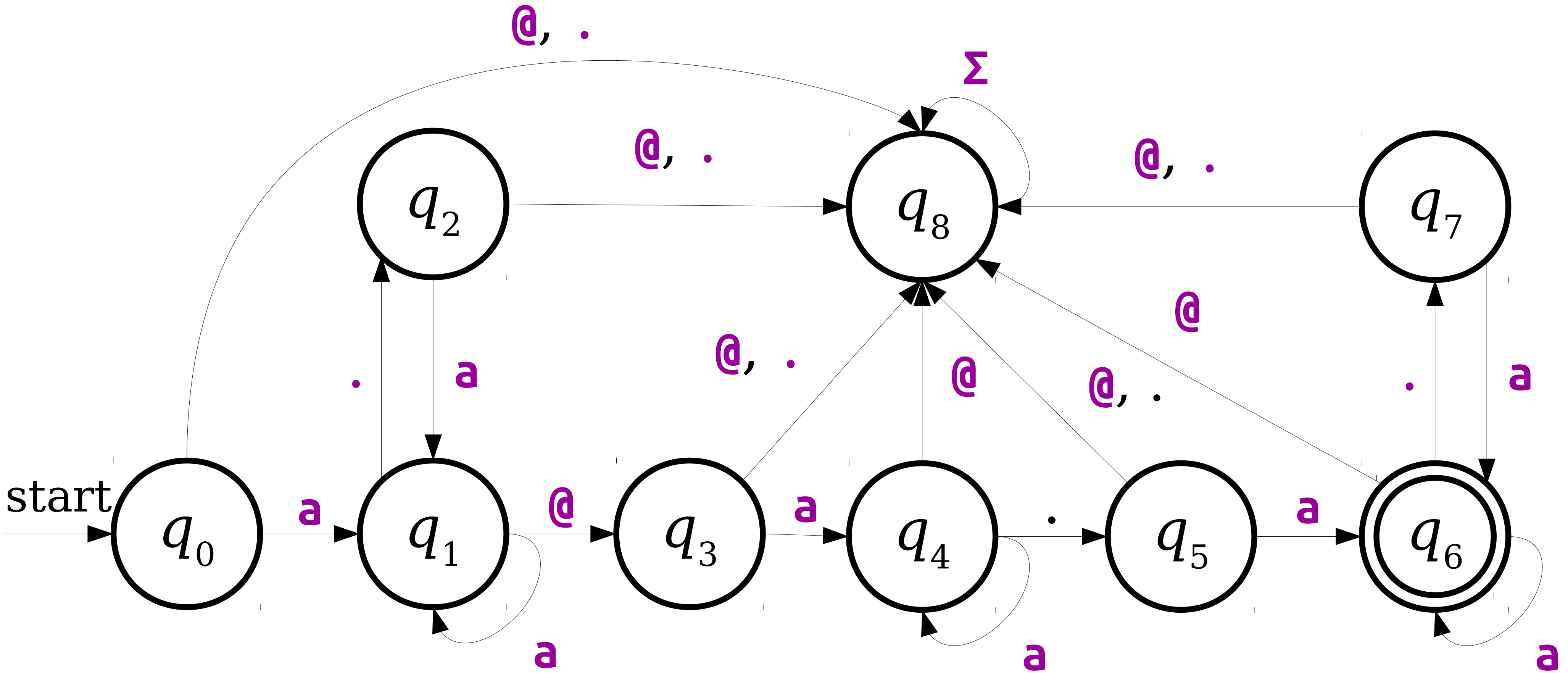
For Comparison

a⁺(.a⁺)*@a⁺(.a⁺)⁺



Shorthand Summary

- R^n is shorthand for $RR \dots R$ (n times).
  - Edge case: define $R^0 = \varepsilon$.
- Σ is shorthand for "any character in Σ."
- **R?** is shorthand for **(R ∪ ε)**, meaning "zero or one copies of *R*."
- R^+ is shorthand for **RR***, meaning "one or more copies of *R*."

Time-Out for Announcements!

Problem Sets

- Problem Set Four was due at 3:00PM today.
- Problem Set Five goes out today. It’s due next Friday at 3:00PM.
  - Play around with DFAs, NFAs, regular expressions, and their properties!
  - Explore how all the discrete math topics we’ve talked about so far come into play!

“Practice Midterm” Exam

- We’ve released a completely optional “practice midterm” exam composed of what we think is a good representative sample of older midterm questions from across the years, covering topics from the first half of the course.
- **There is no midterm in this course**, but we recommend taking some time in the next week to actually sit down and try taking this exam to check your understanding of what we’ve covered so far.

Back to CS103!

The Power of Regular Expressions

Theorem: If R is a regular expression, then $\mathcal{L}(R)$ is regular.

Proof idea: Use induction!

- The atomic regular expressions all represent regular languages.
- The combination steps represent closure properties.
- So anything you can make from them must be regular!

Thompson’s Algorithm

- In practice, many regex matchers use an algorithm called ***Thompson's algorithm*** to convert regular expressions into NFAs (and, from there, to DFAs).
  - Read Sipser if you’re curious!
- ***Fun fact:*** the “Thompson” here is Ken Thompson, one of the co-inventors of Unix!

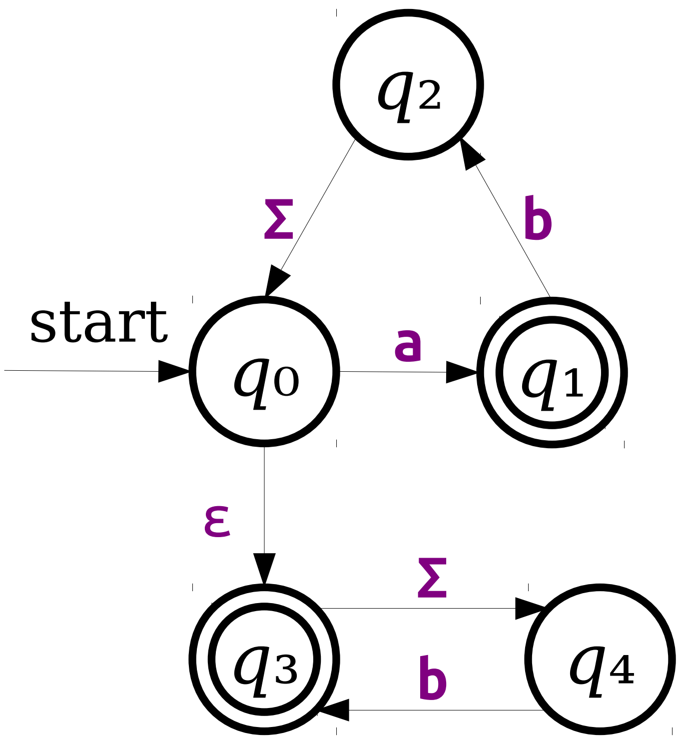
The Power of Regular Expressions

Theorem: If L is a regular language, then there is a regular expression for L.

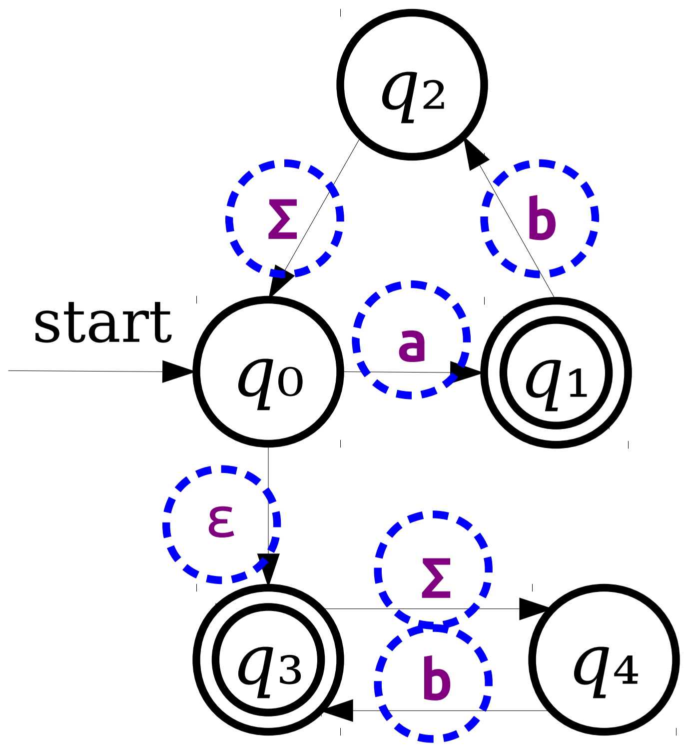
This is not obvious!

Proof idea: Show how to convert an arbitrary NFA into a regular expression.

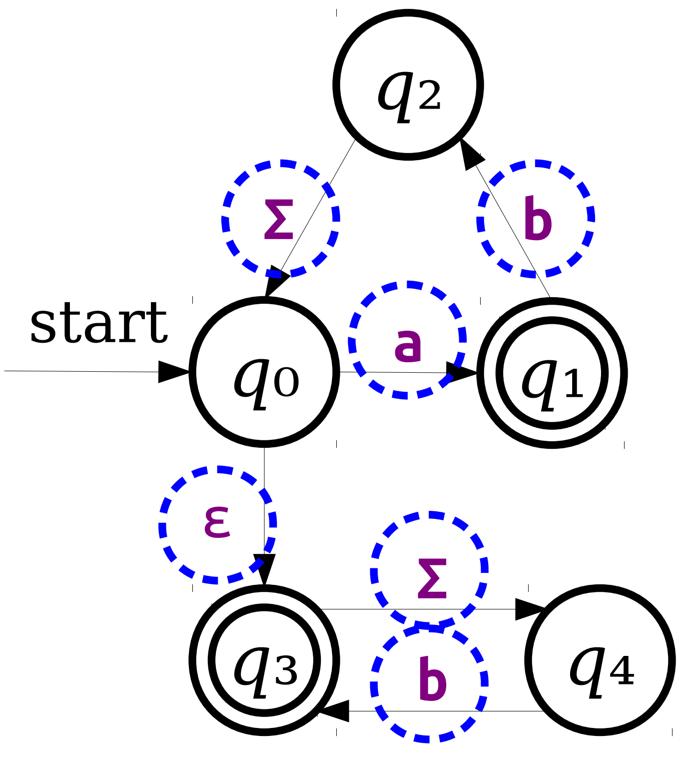
Generalizing NFAs



Generalizing NFAs

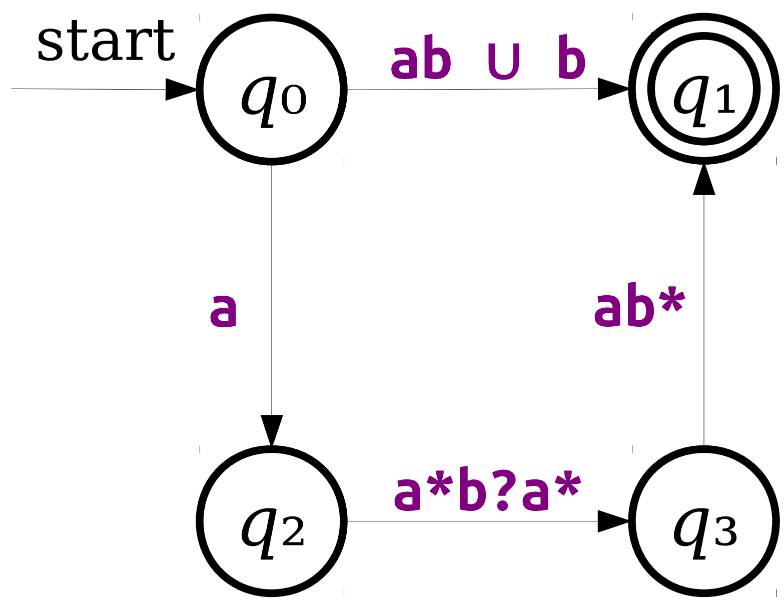


Generalizing NFAs



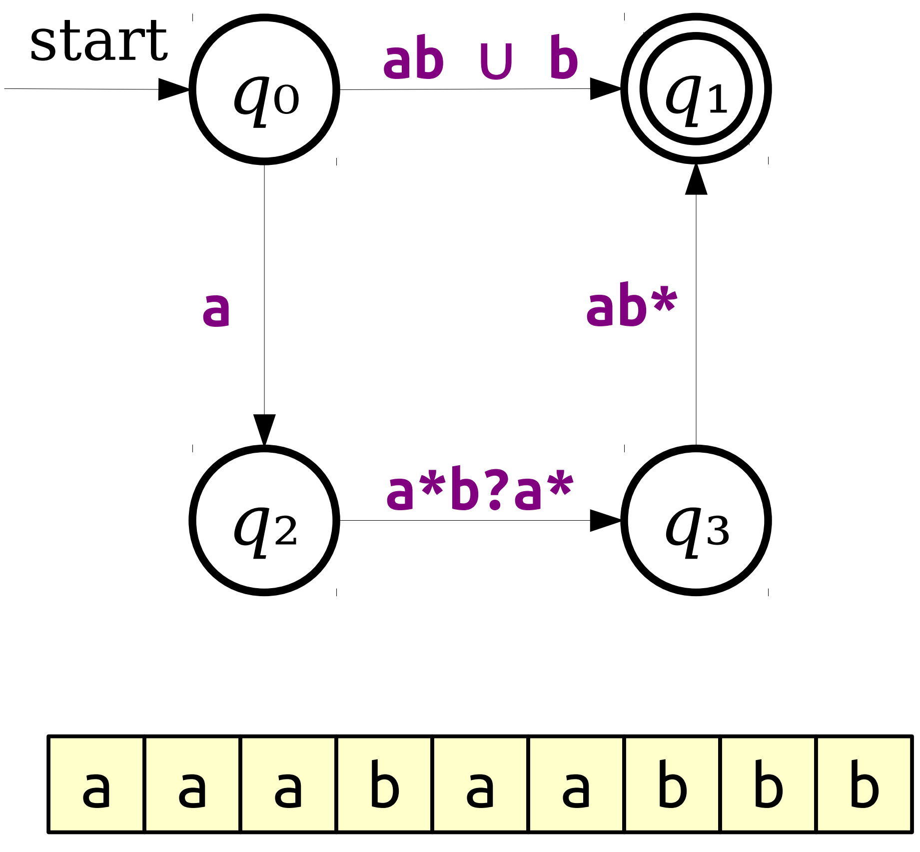
These are all regular expressions!

Generalizing NFAs

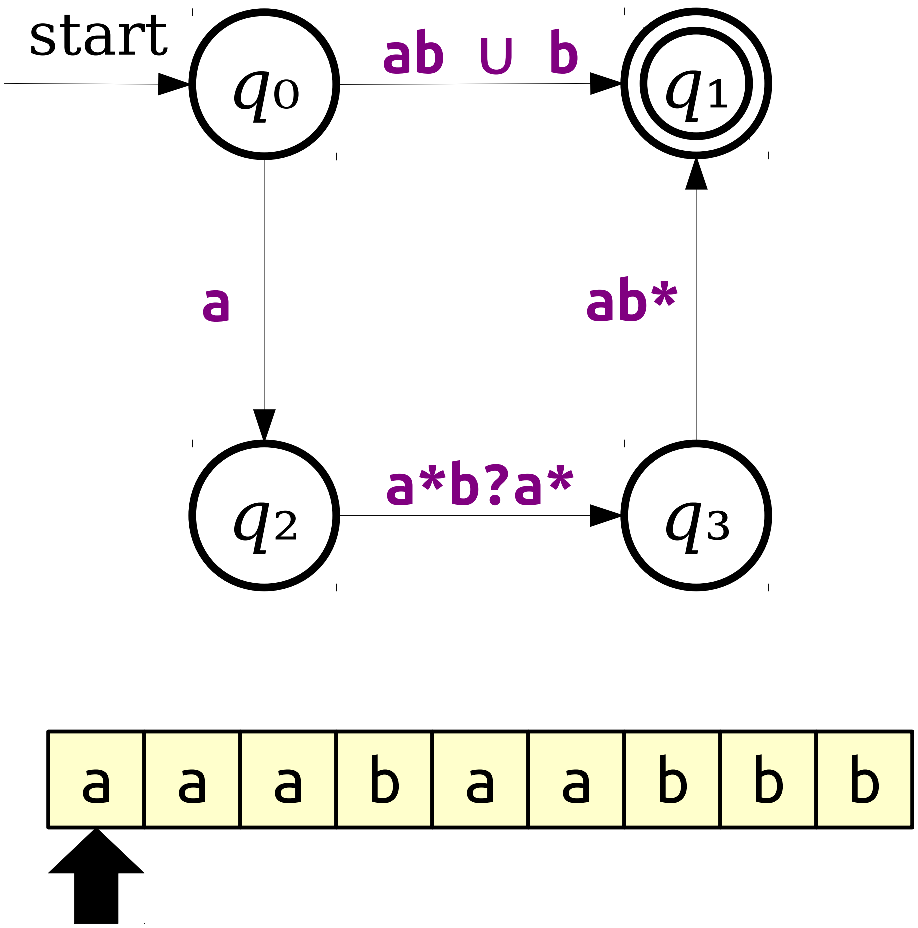


Note: Actual NFAs aren't allowed to have transitions like these. This is just a thought experiment.

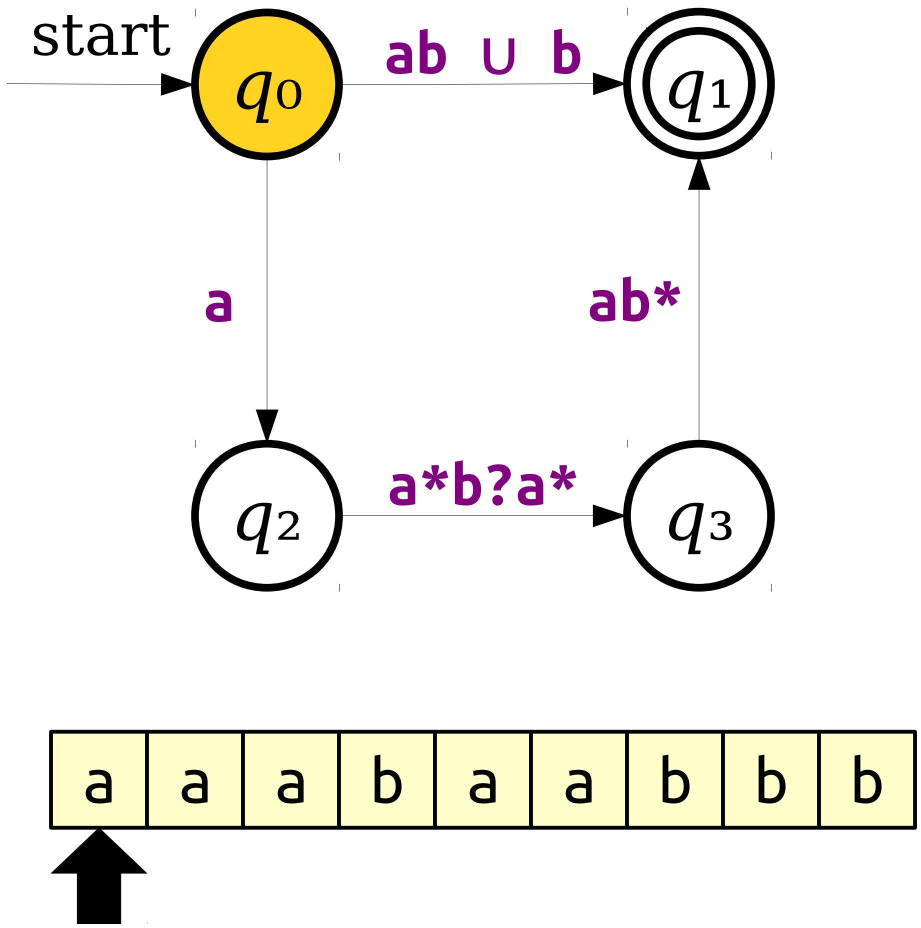
Generalizing NFAs



Generalizing NFAs



Generalizing NFAs



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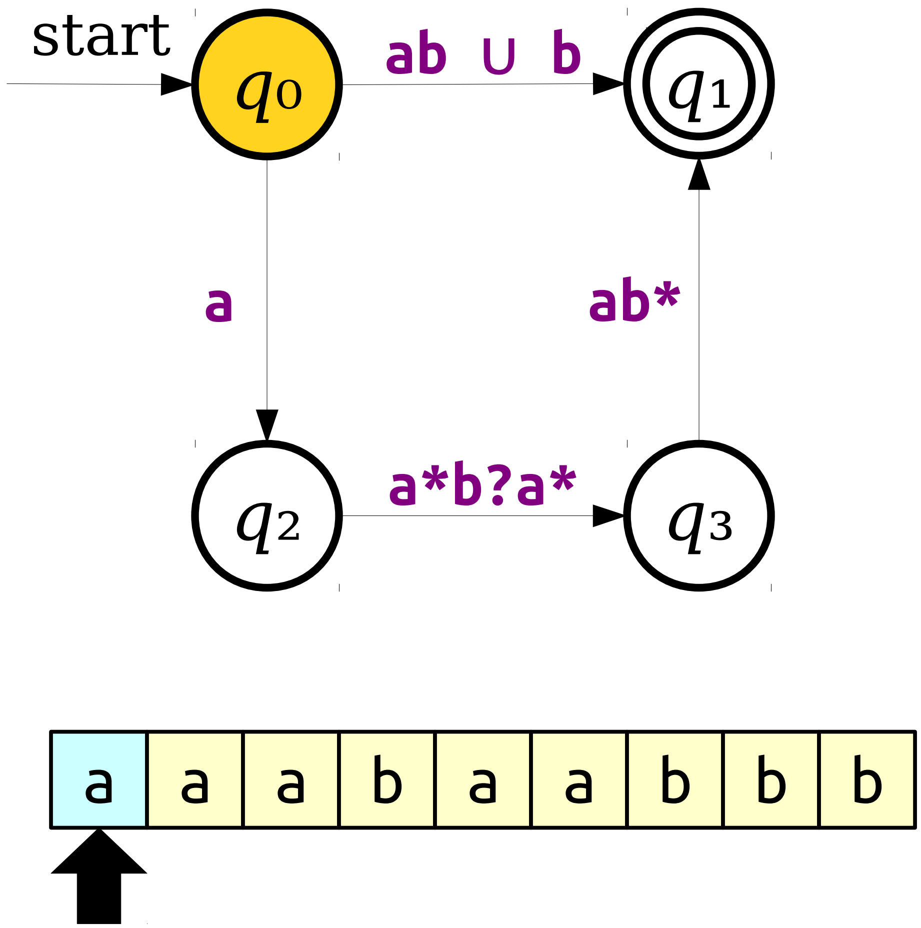
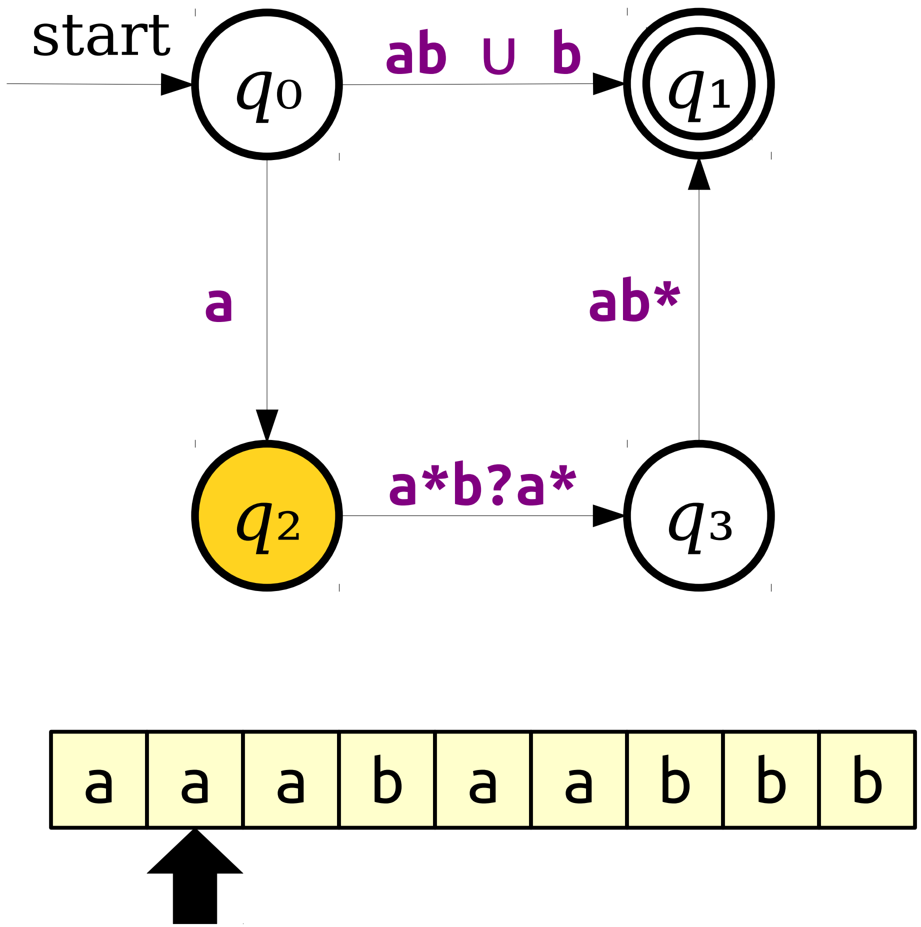
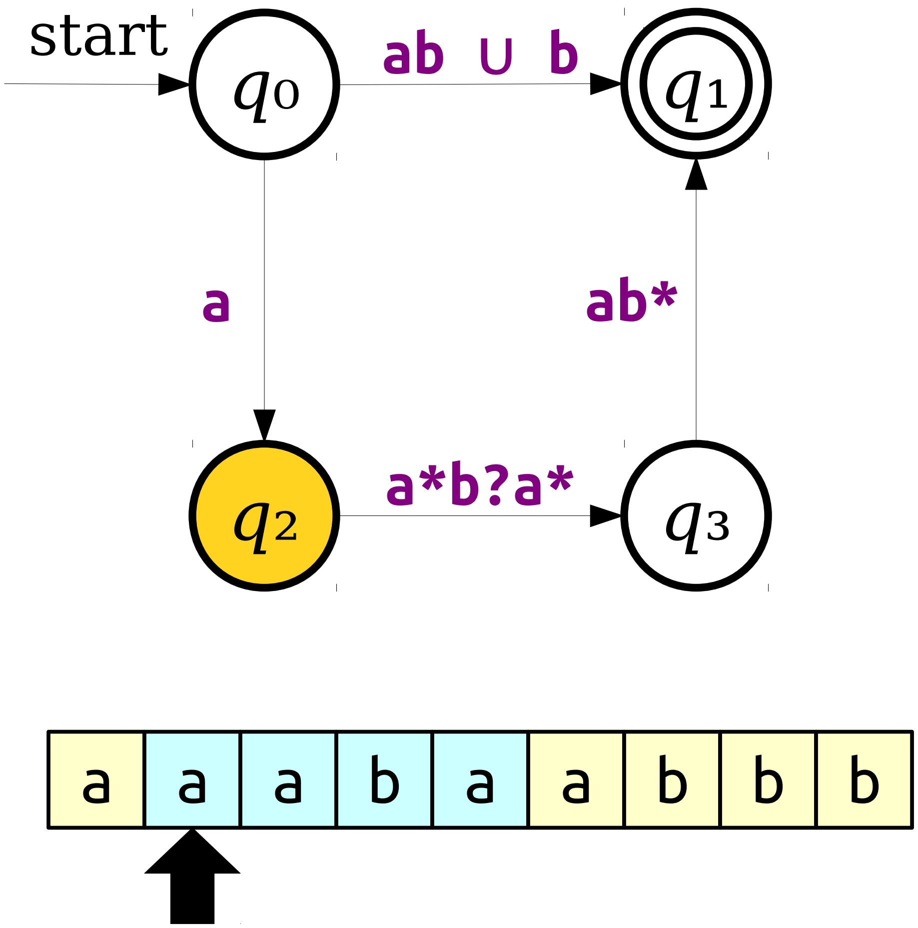


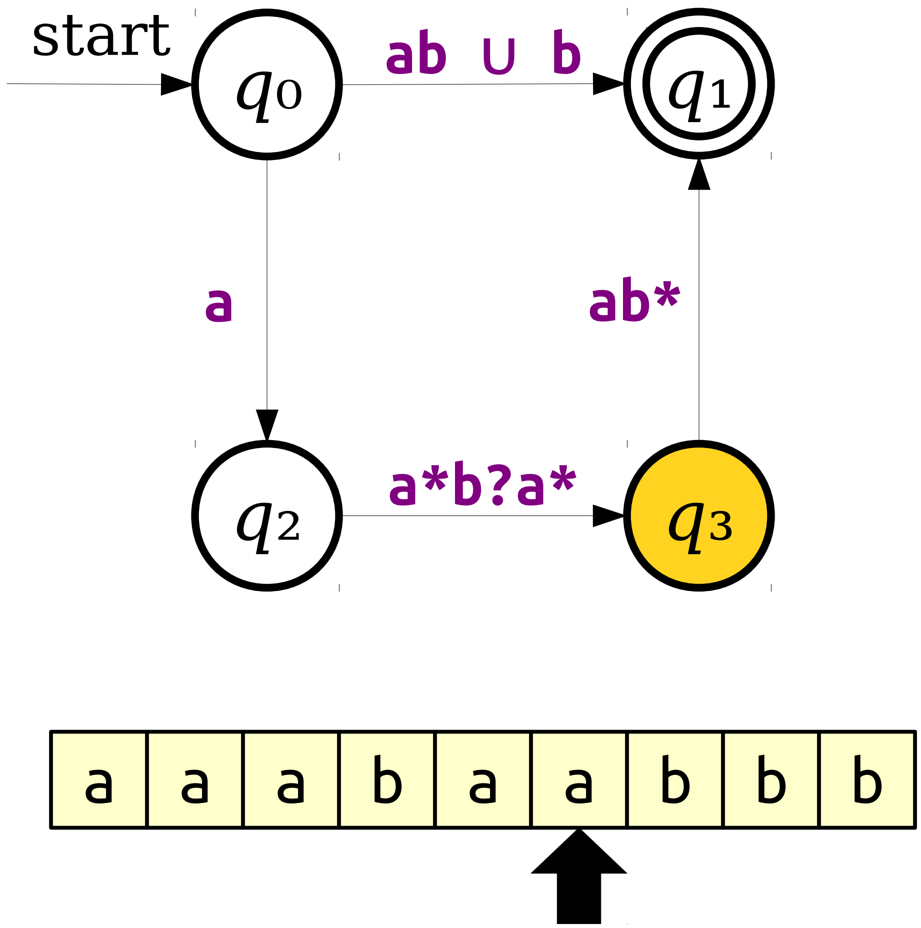
Generalizing NFAs



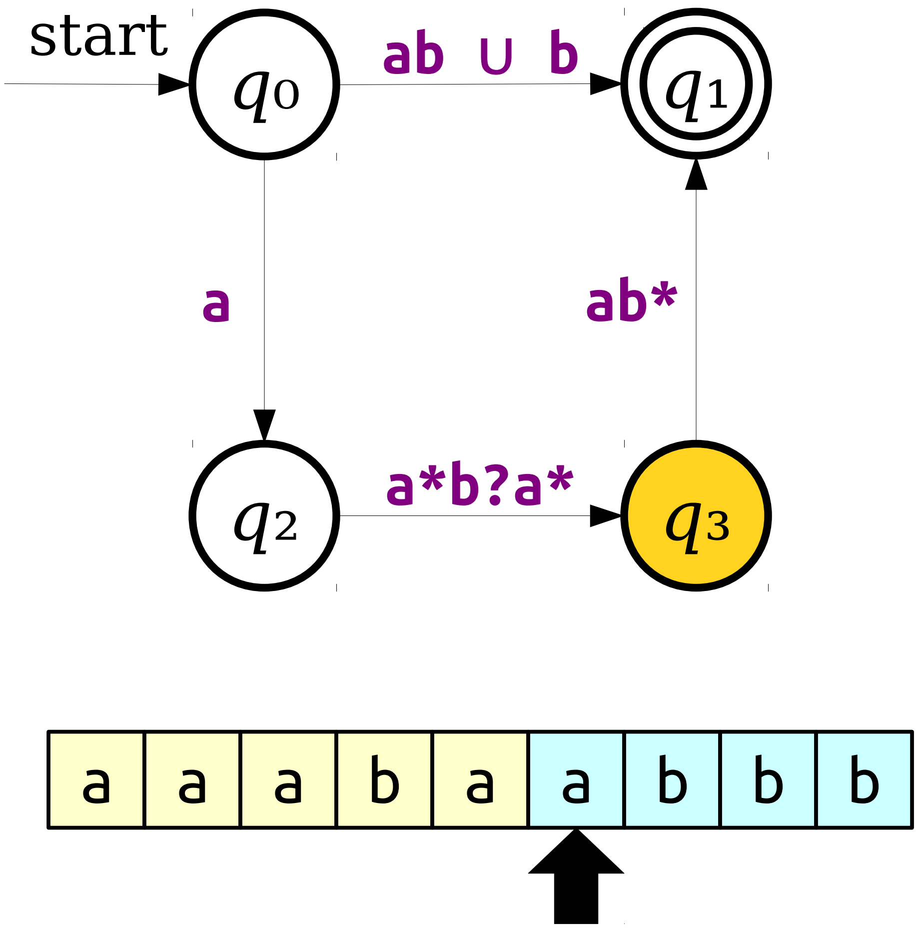
Generalizing NFAs



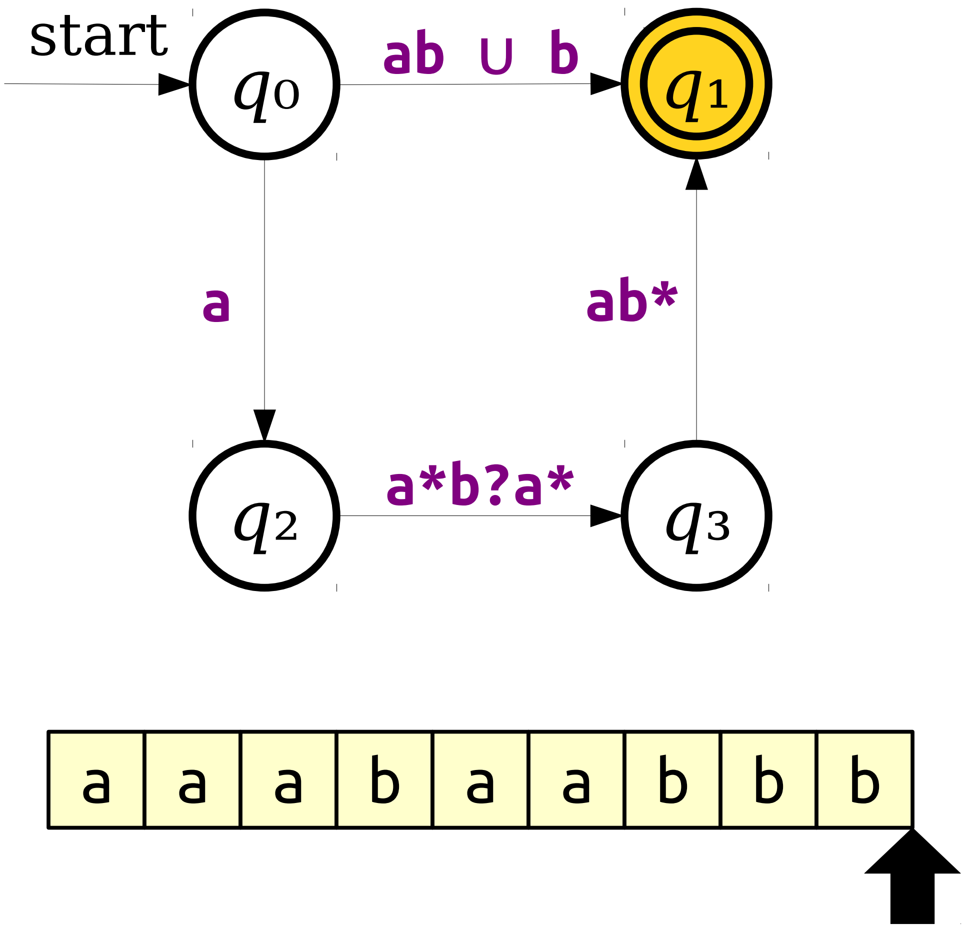
Generalizing NFAs



Generalizing NFAs



Generalizing NFAs



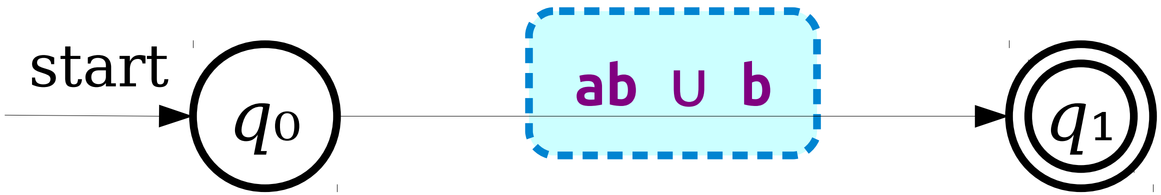
Key Idea 1: Imagine that we can label transitions in an NFA with arbitrary regular expressions.

Generalizing NFAs

Generalizing NFAs

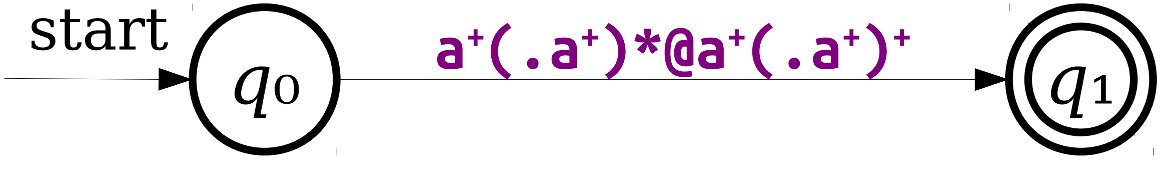
Is there a simple regular expression for the language of this generalized NFA?

Generalizing NFAs

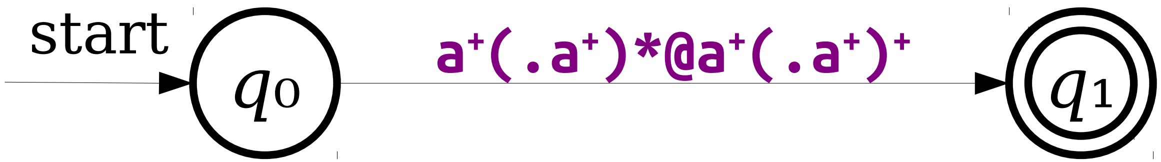


Is there a simple regular expression for the language of this generalized NFA?

Generalizing NFAs

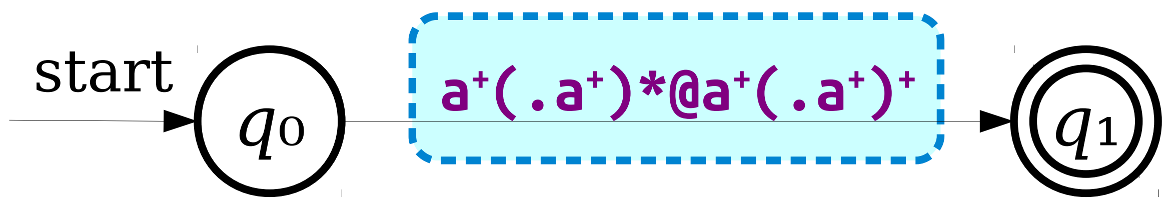


Generalizing NFAs



Is there a simple regular expression for the language of this generalized NFA?

Generalizing NFAs

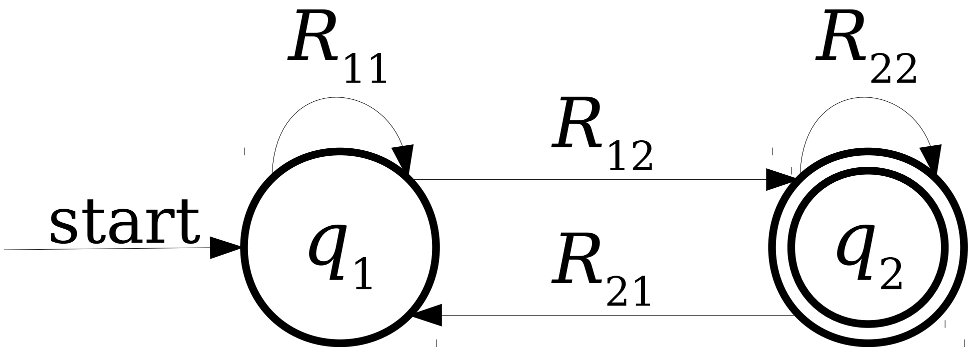


Is there a simple regular expression for the language of this generalized NFA?

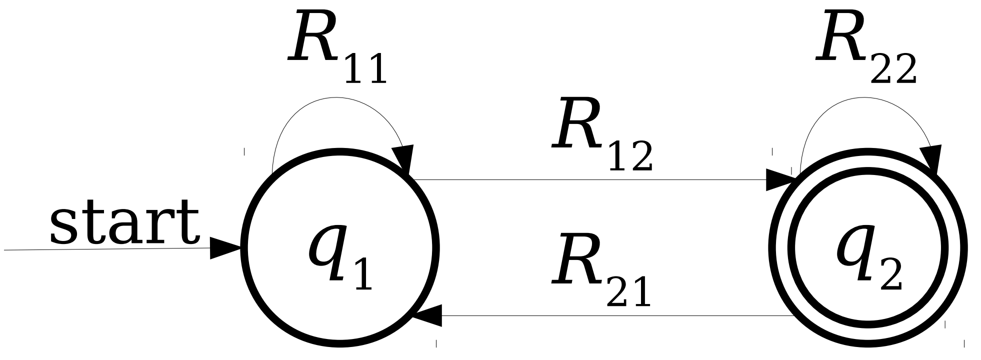
Key Idea 2: If we can convert an NFA into a generalized NFA that looks like this...

...then we can easily read off a regular expression for the original NFA.

From NFAs to Regular Expressions

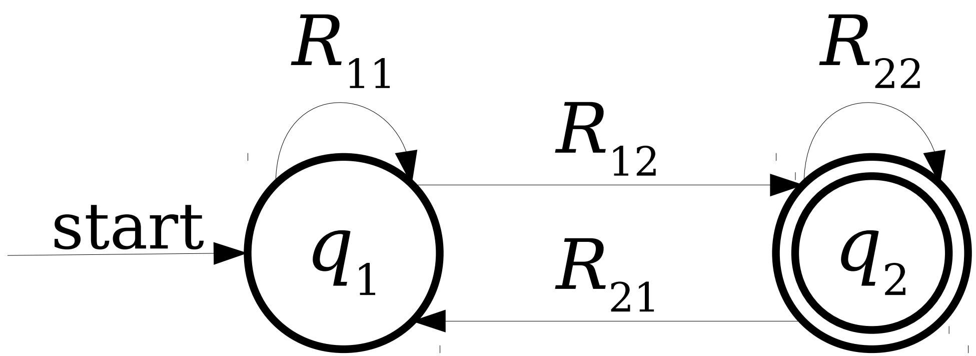


From NFAs to Regular Expressions



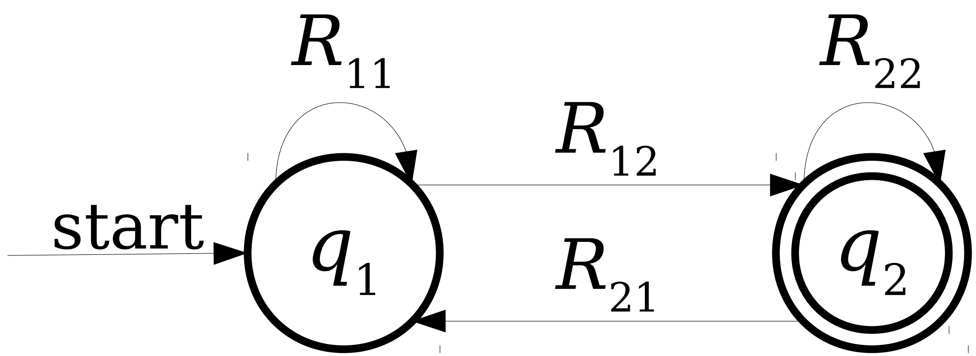
Here, R_{11}, R_{12}, R_{21}, and R_{22} are arbitrary regular expressions.

From NFAs to Regular Expressions



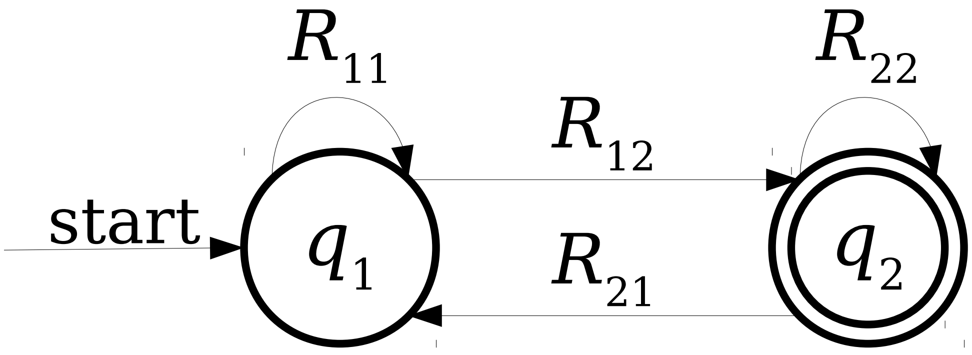
Question: Can we get a clean regular expression from this NFA?

From NFAs to Regular Expressions



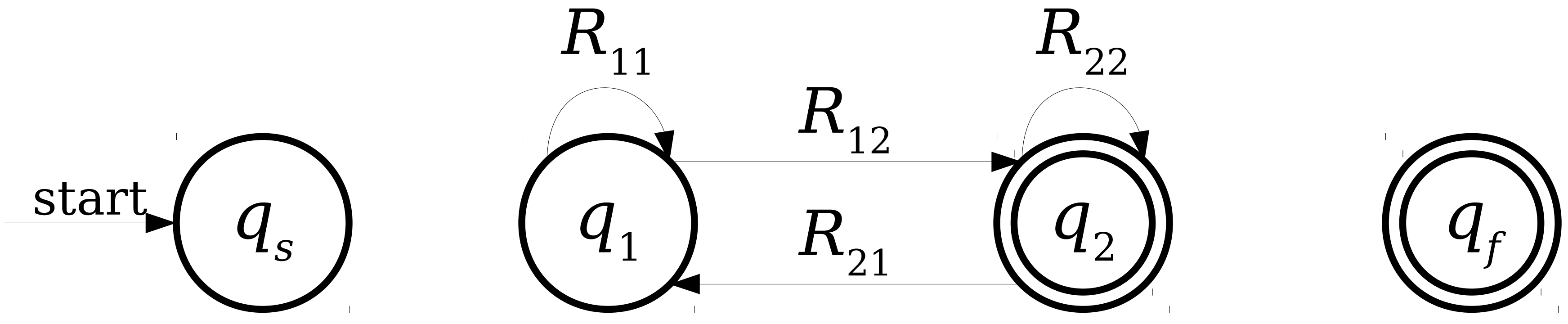
Key Idea 3: Somehow transform this NFA so that it looks like

From NFAs to Regular Expressions

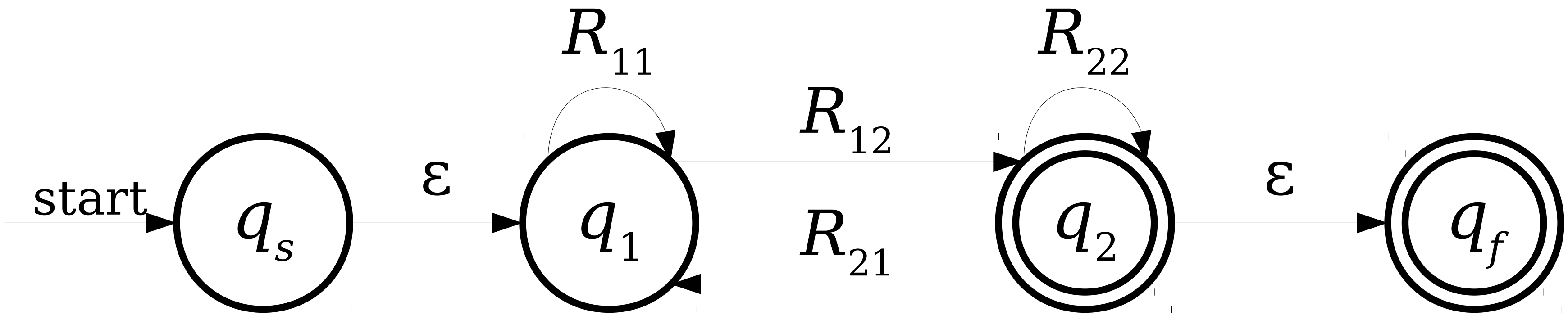


The first step is going to be a bit weird…

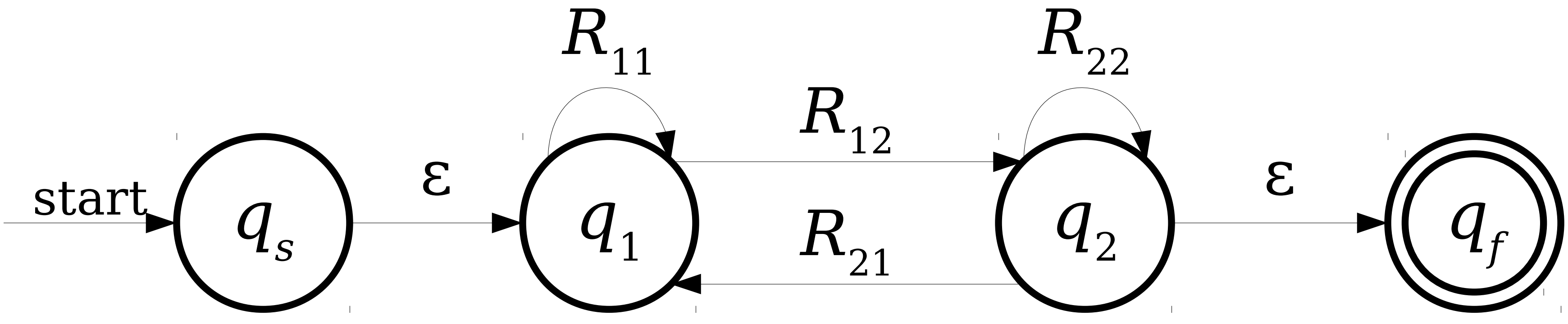
From NFAs to Regular Expressions



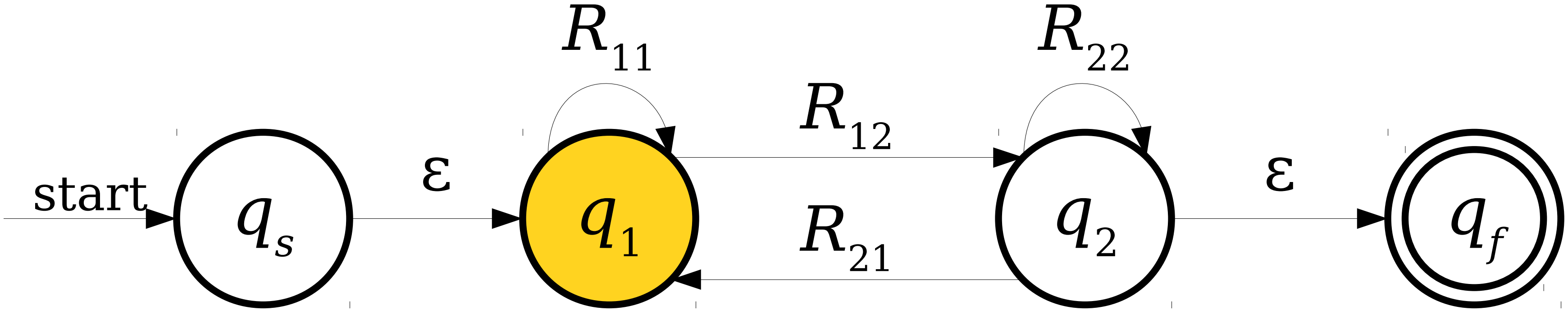
From NFAs to Regular Expressions



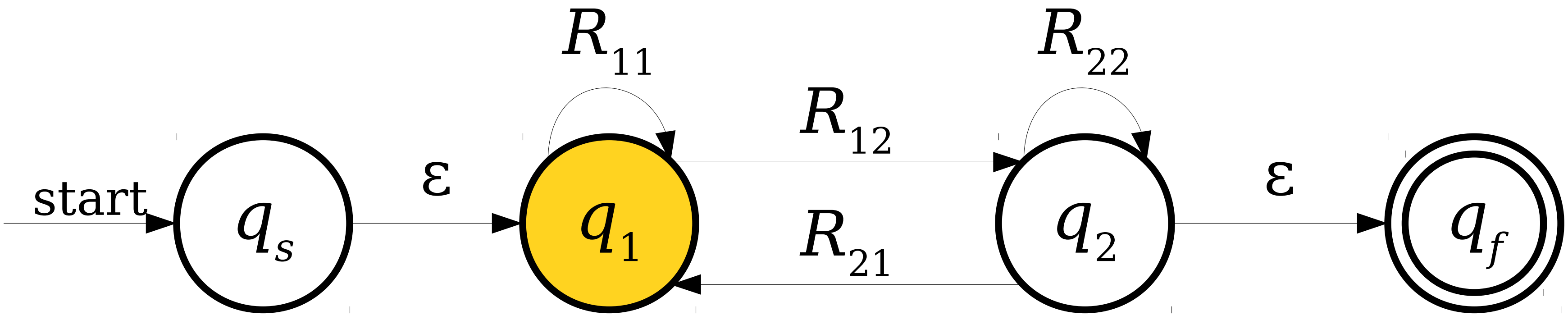
From NFAs to Regular Expressions



From NFAs to Regular Expressions

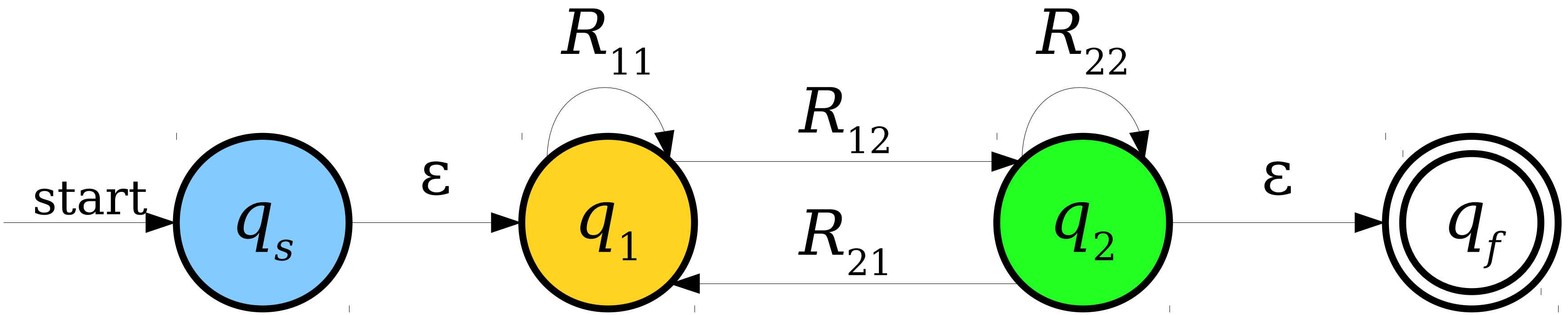


From NFAs to Regular Expressions



Could we eliminate this state from the NFA?

From NFAs to Regular Expressions



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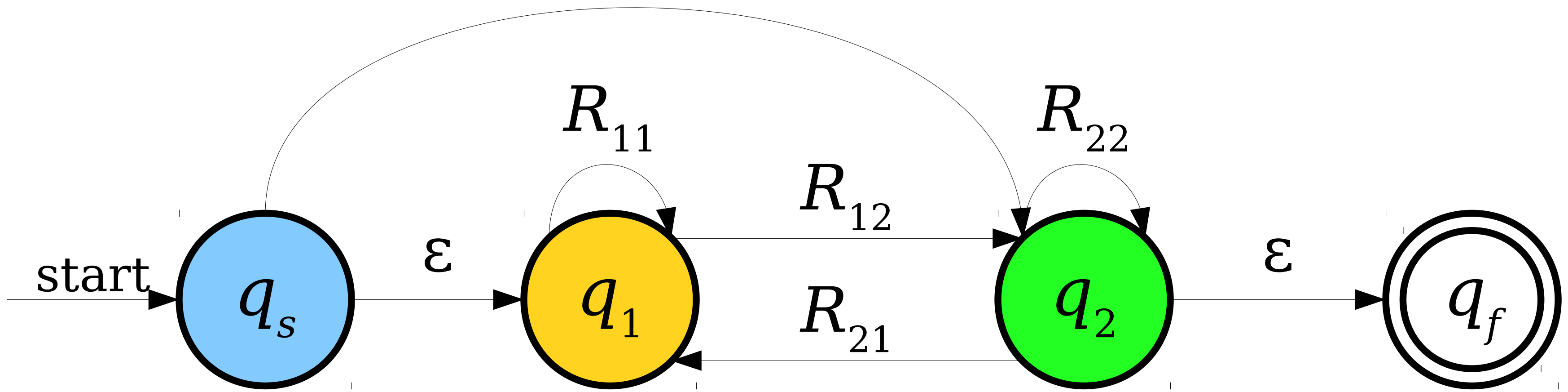

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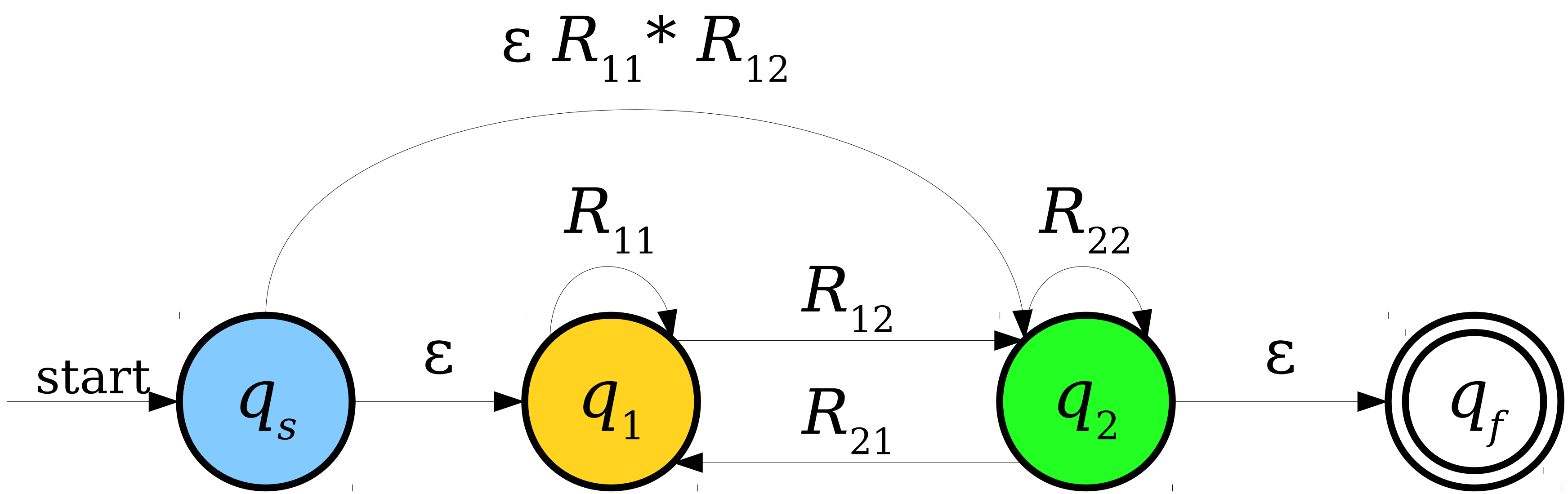
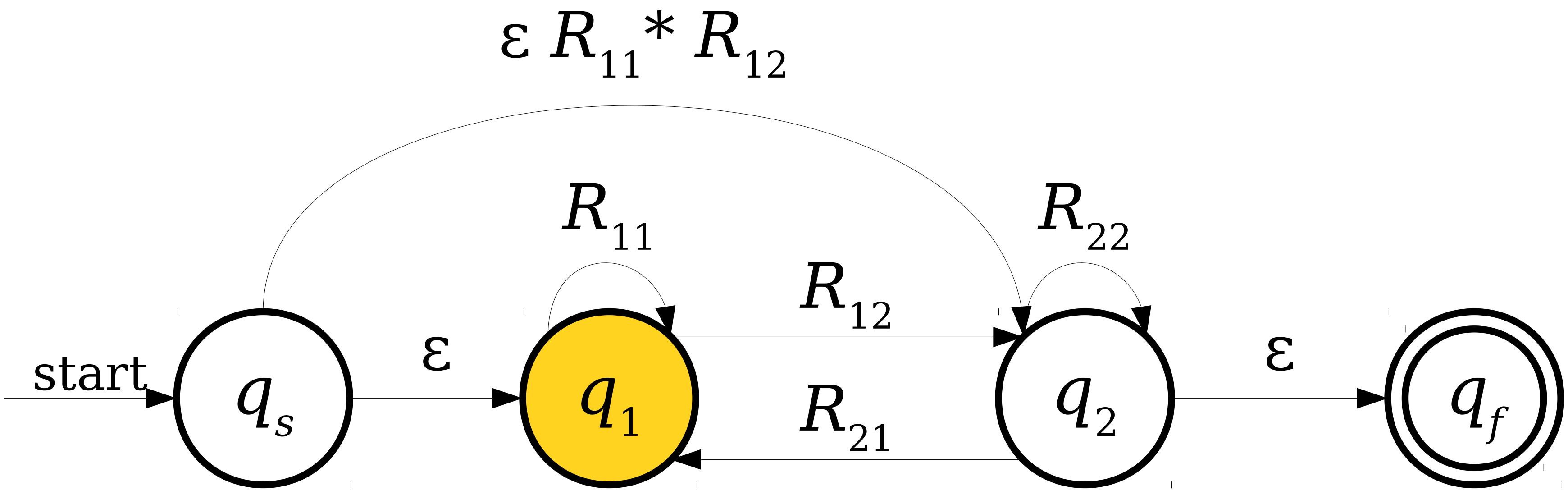
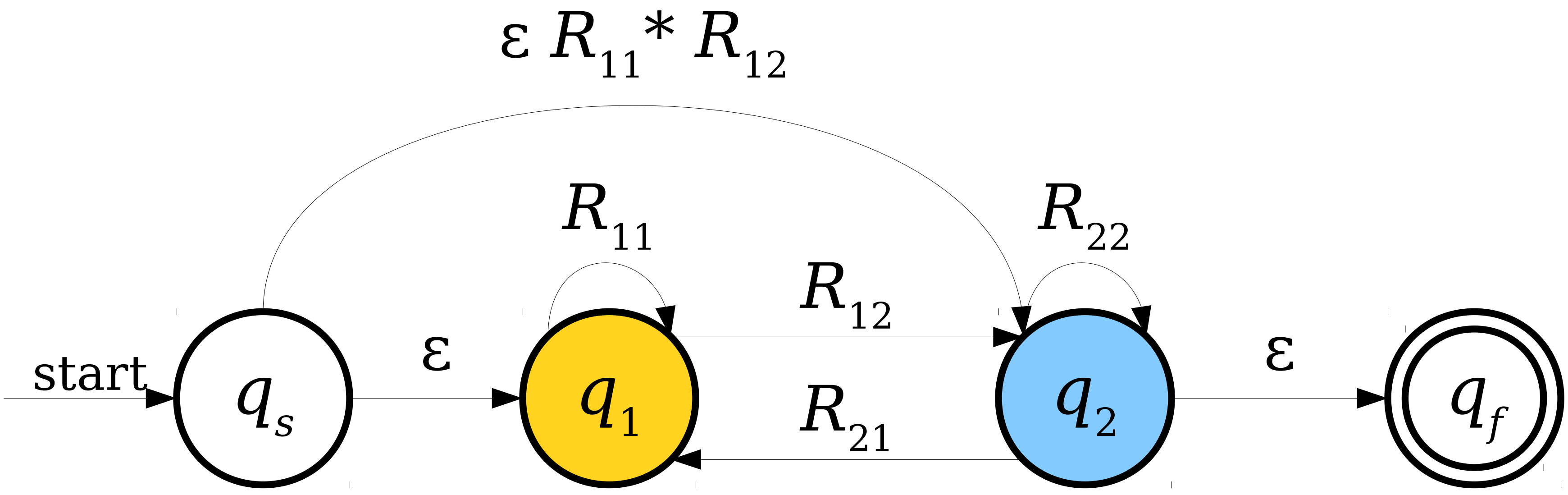


Note: We're using concatenation and Kleene closure in order to skip this state.

From NFAs to Regular Expressions



From NFAs to Regular Expressions



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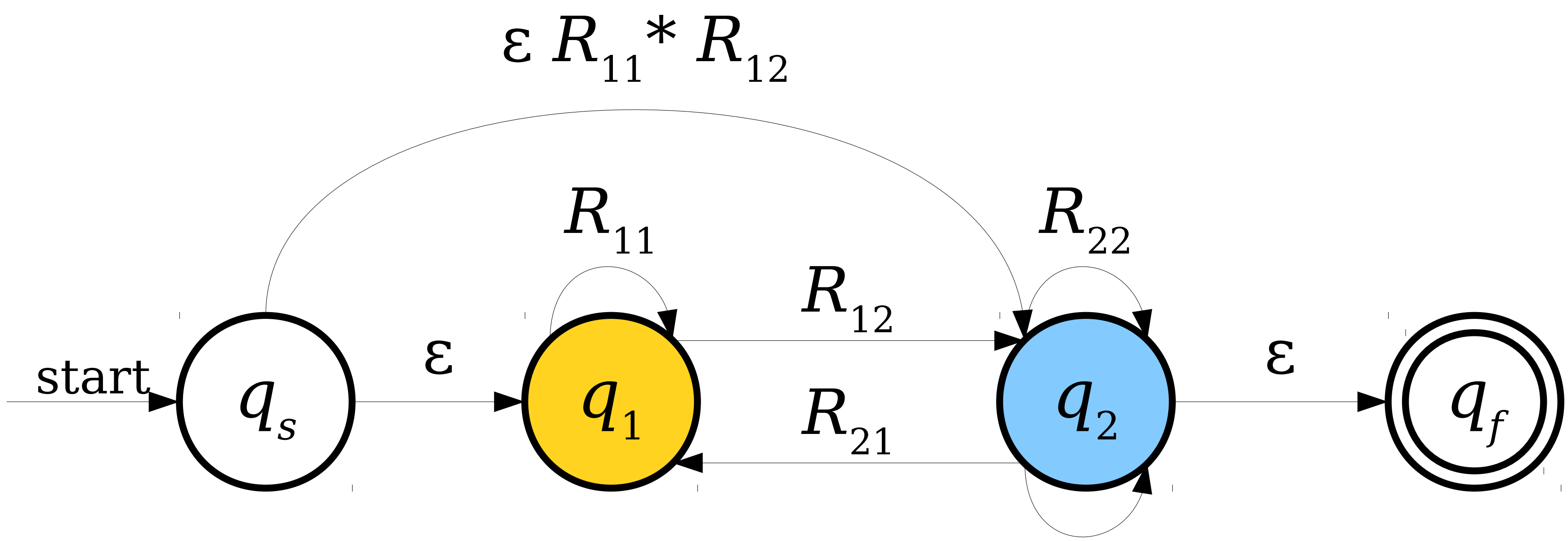
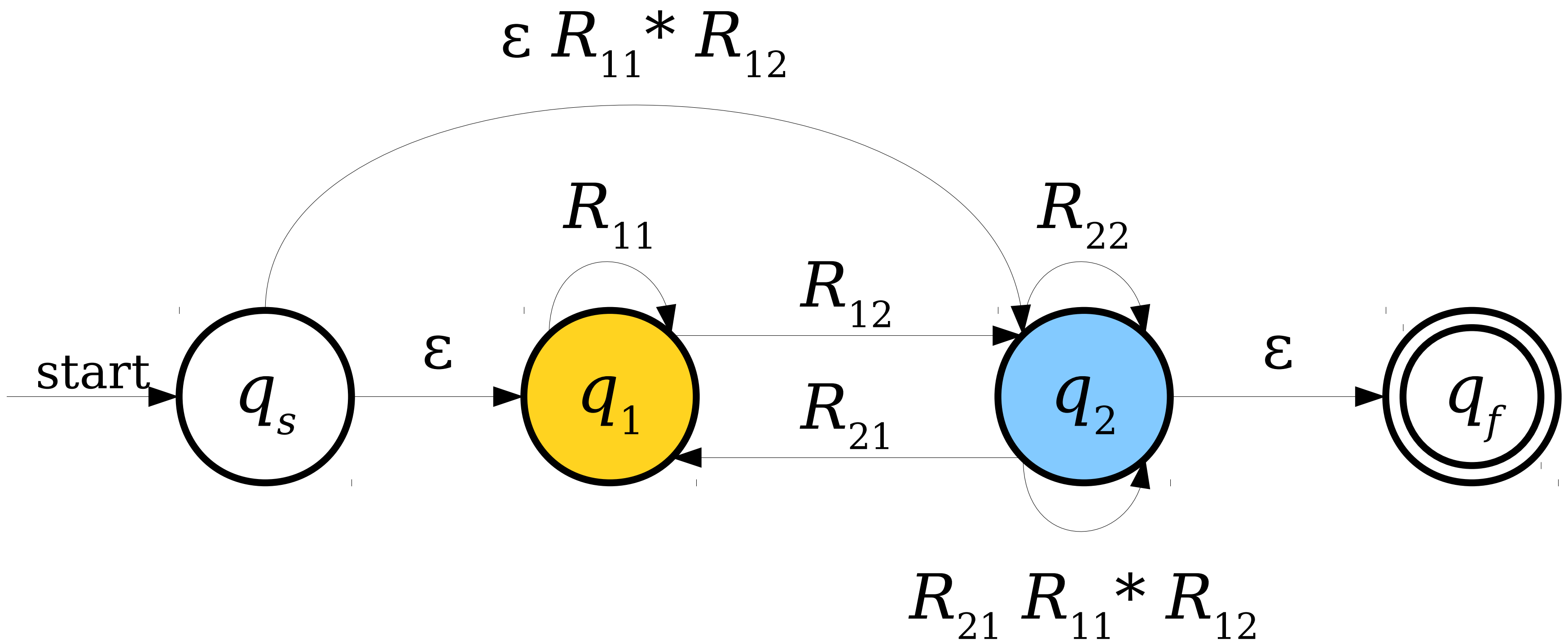
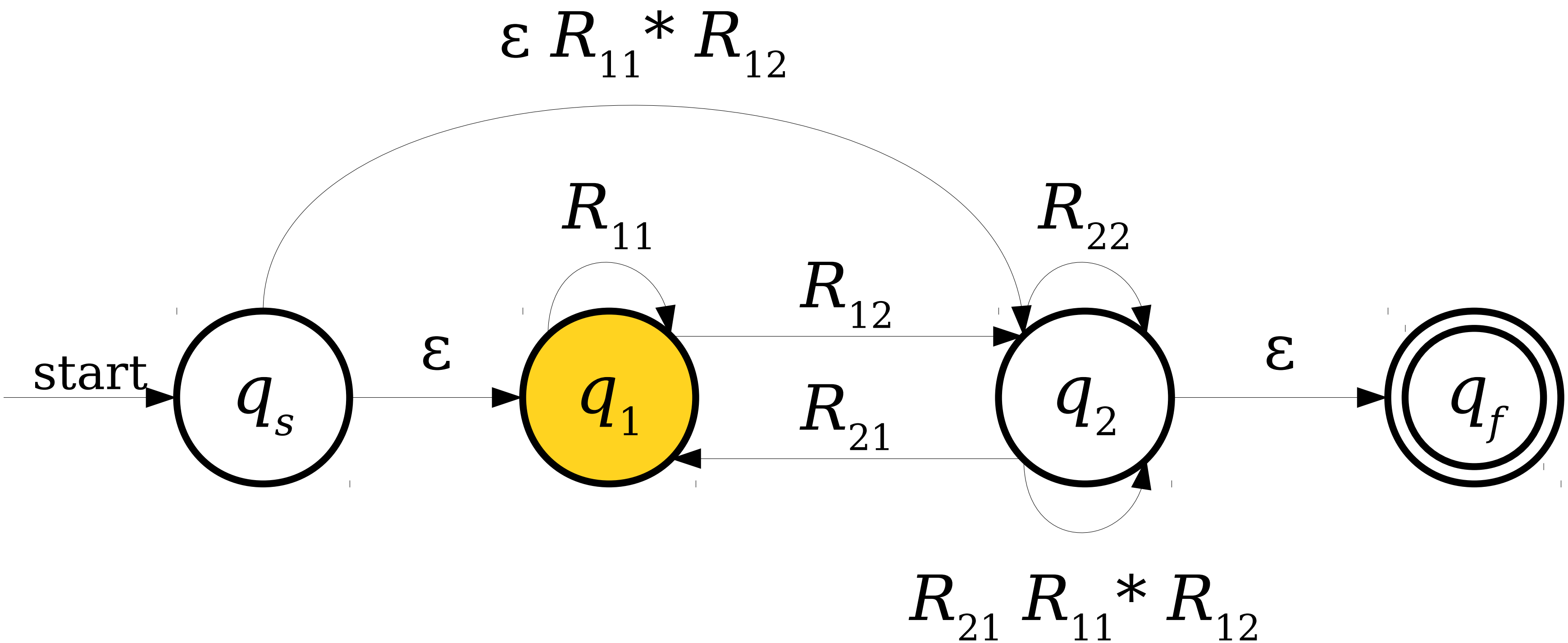


From NFAs to Regular Expressions



From NFAs to Regular Expressions



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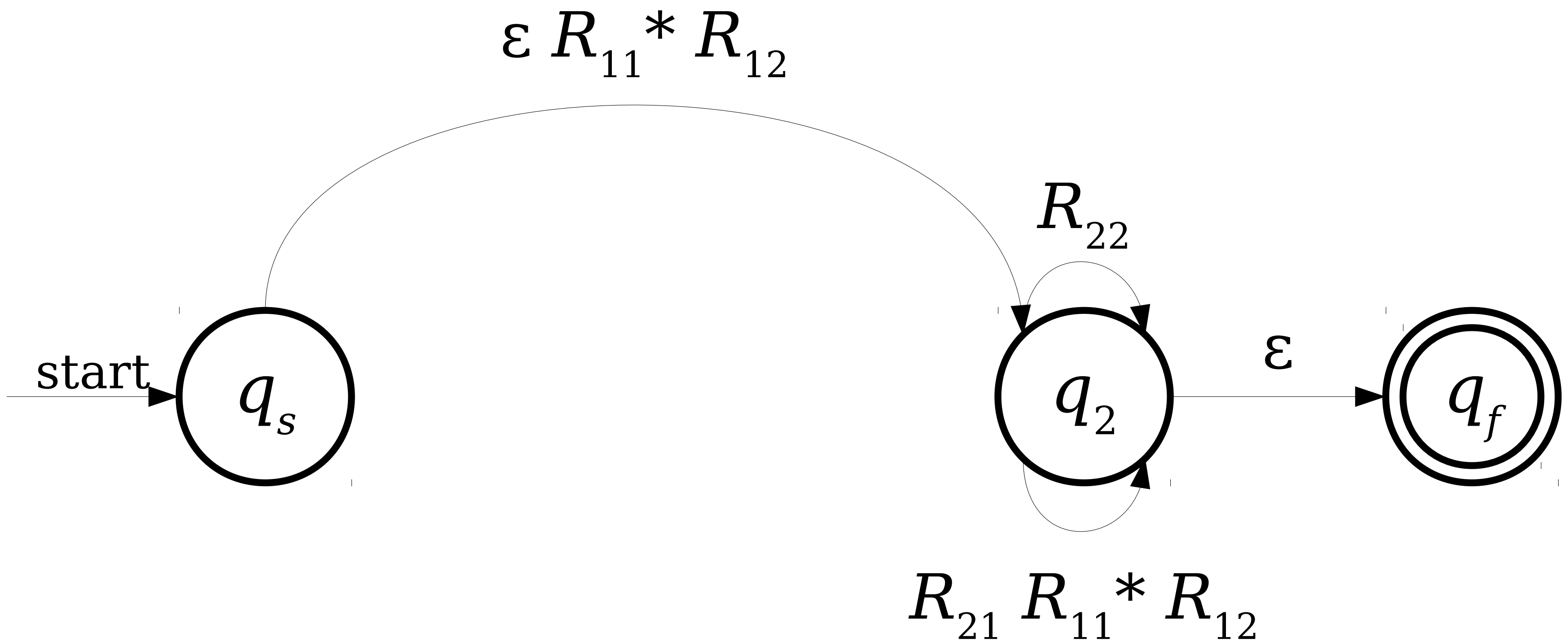
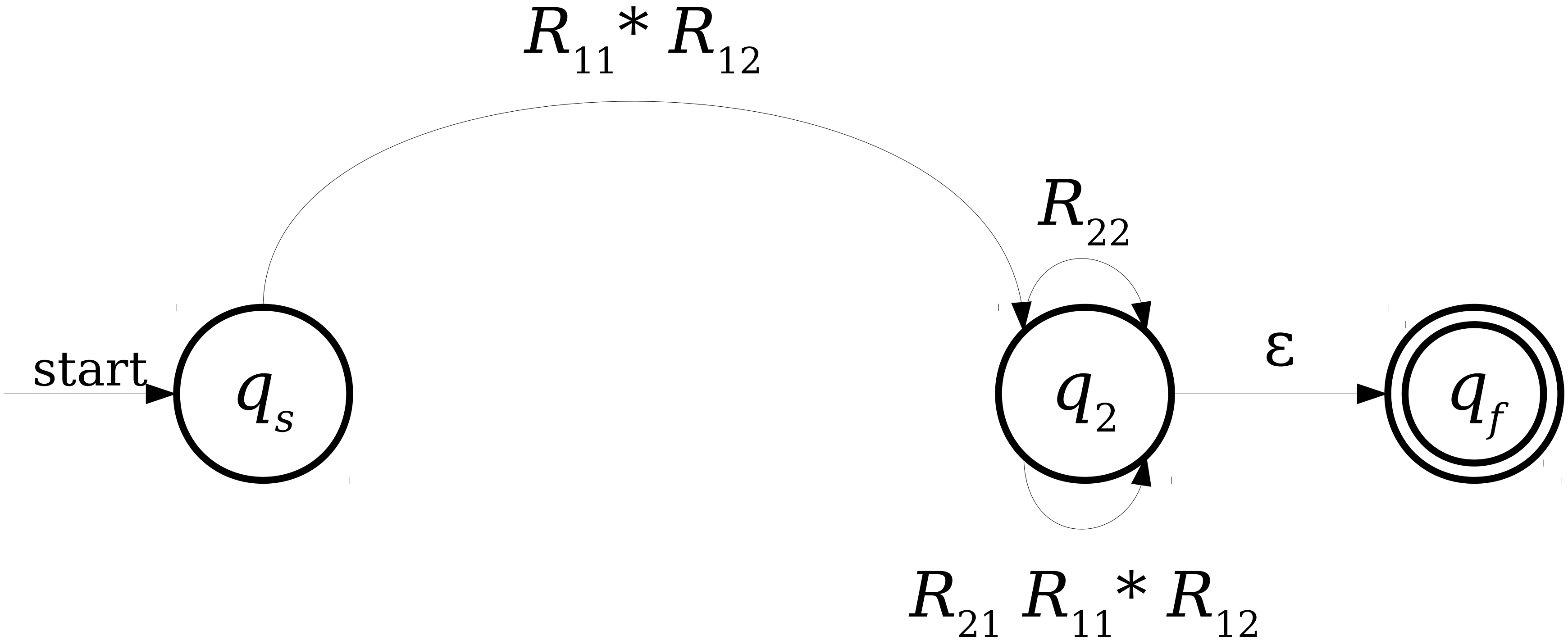
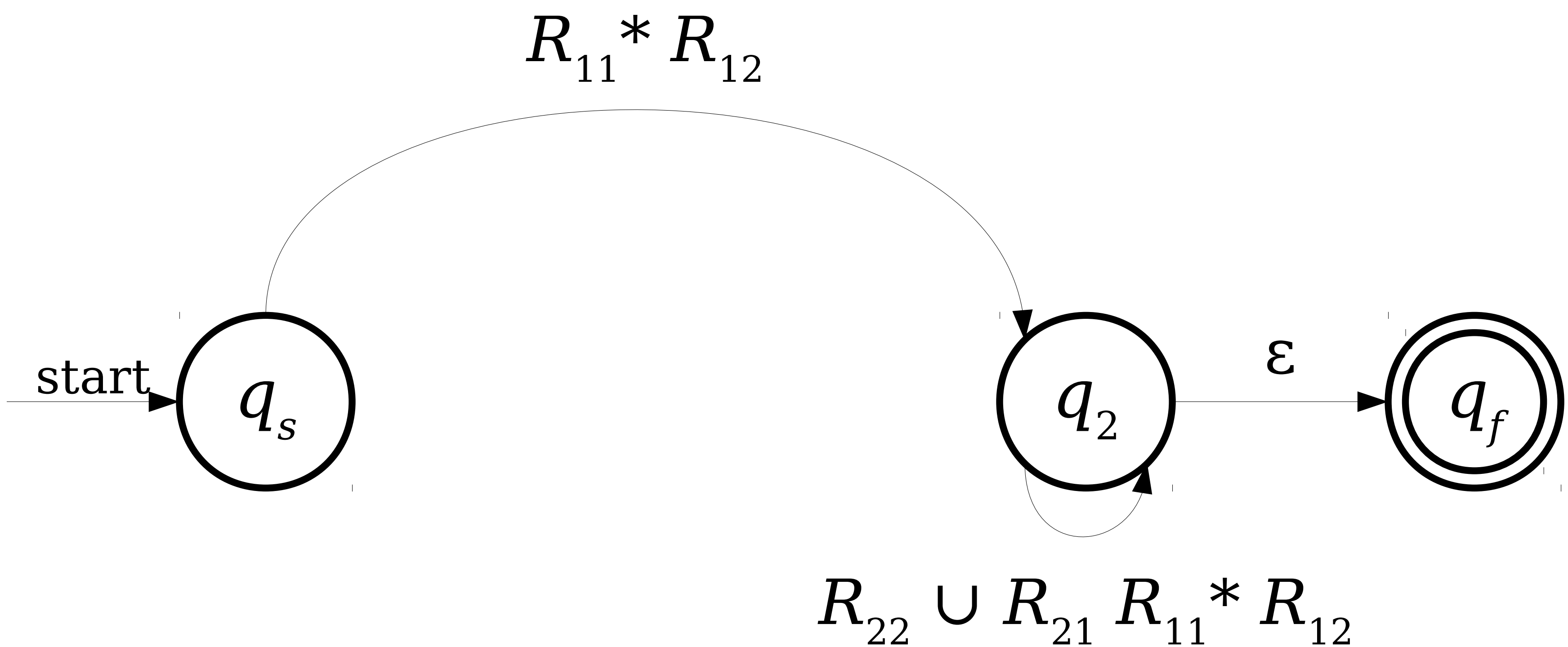


From NFAs to Regular Expressions

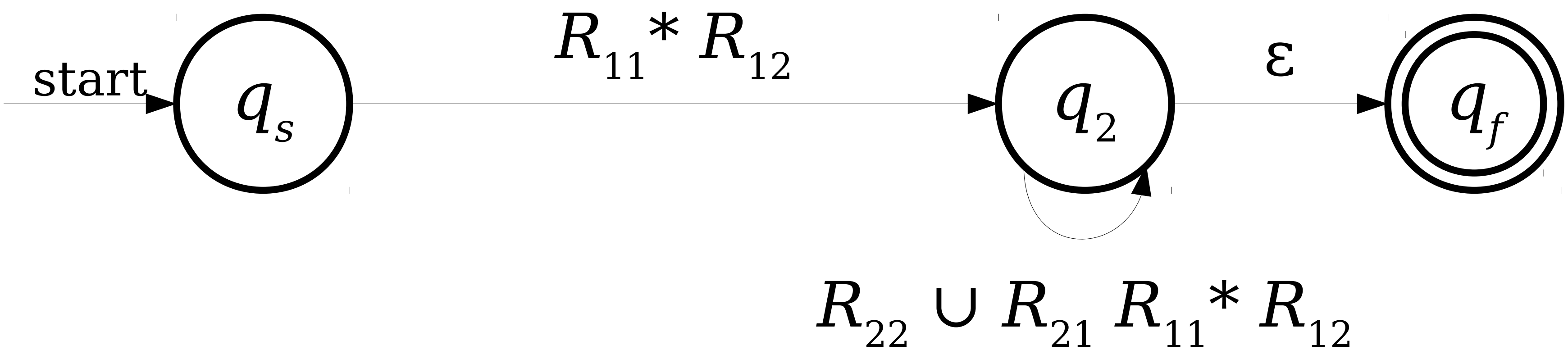


From NFAs to Regular Expressions

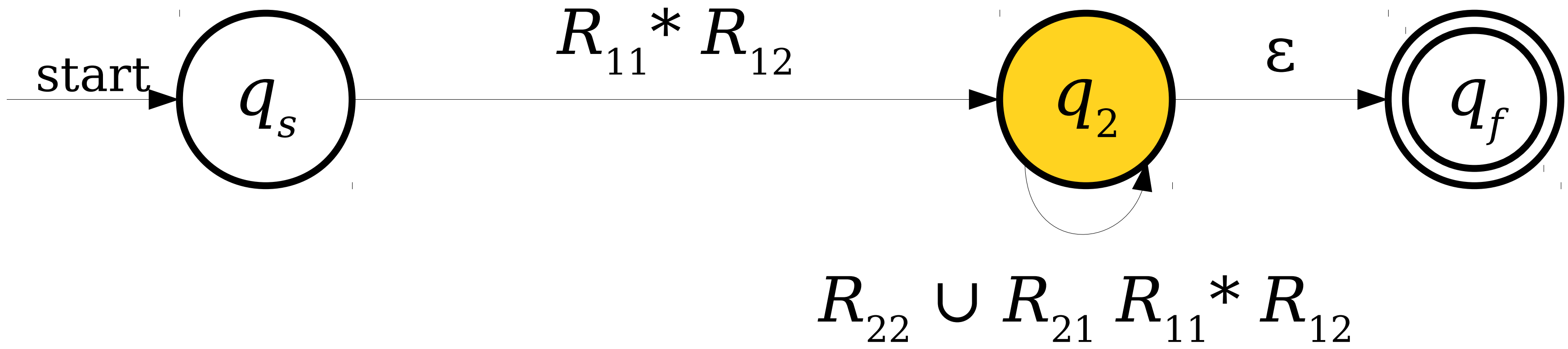


Note: We're using union to combine these transitions together.

From NFAs to Regular Expressions



From NFAs to Regular Expressions



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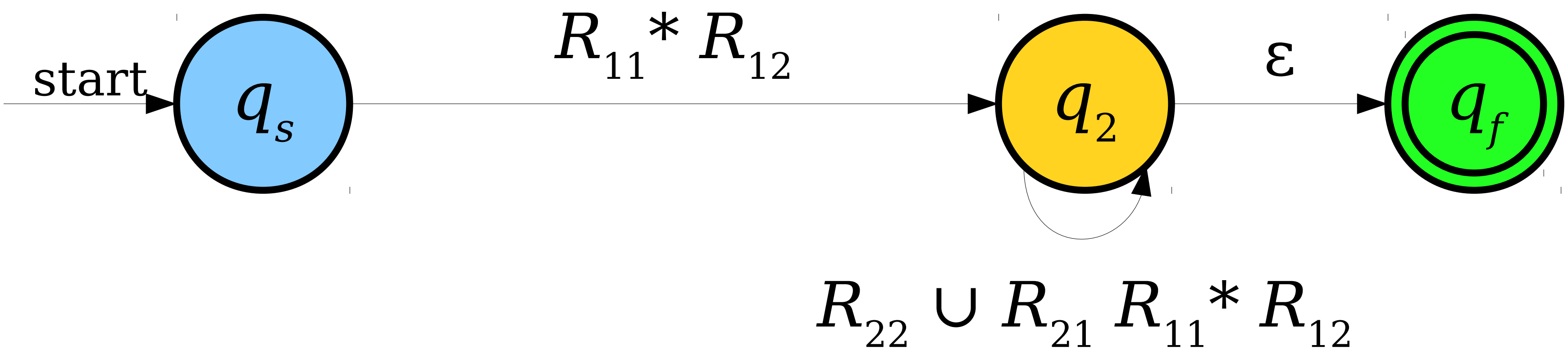

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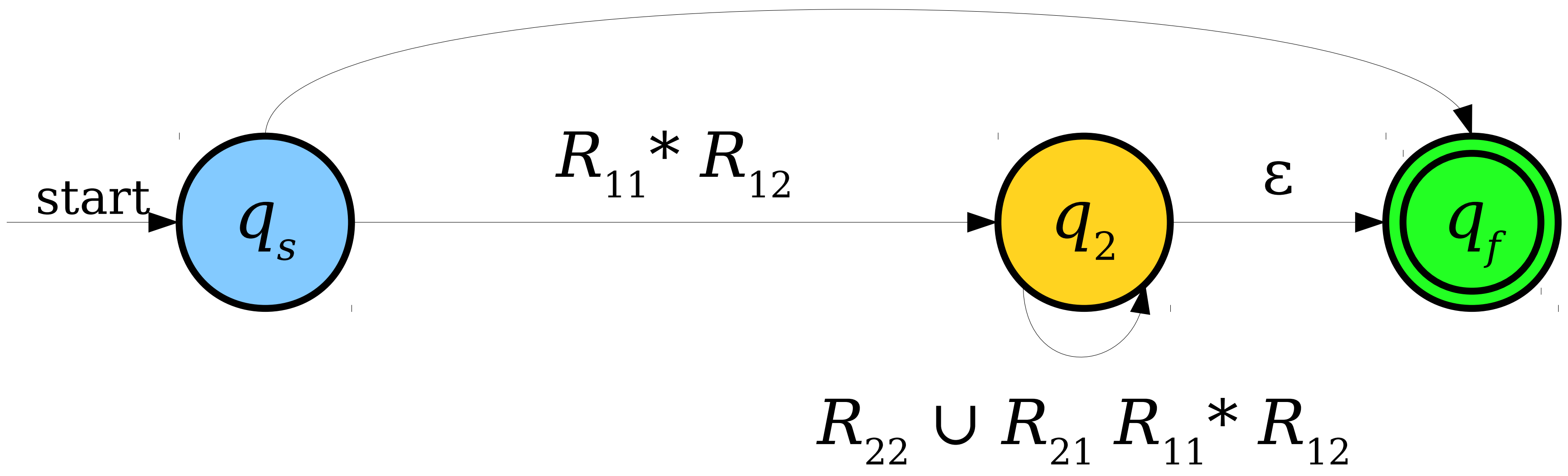

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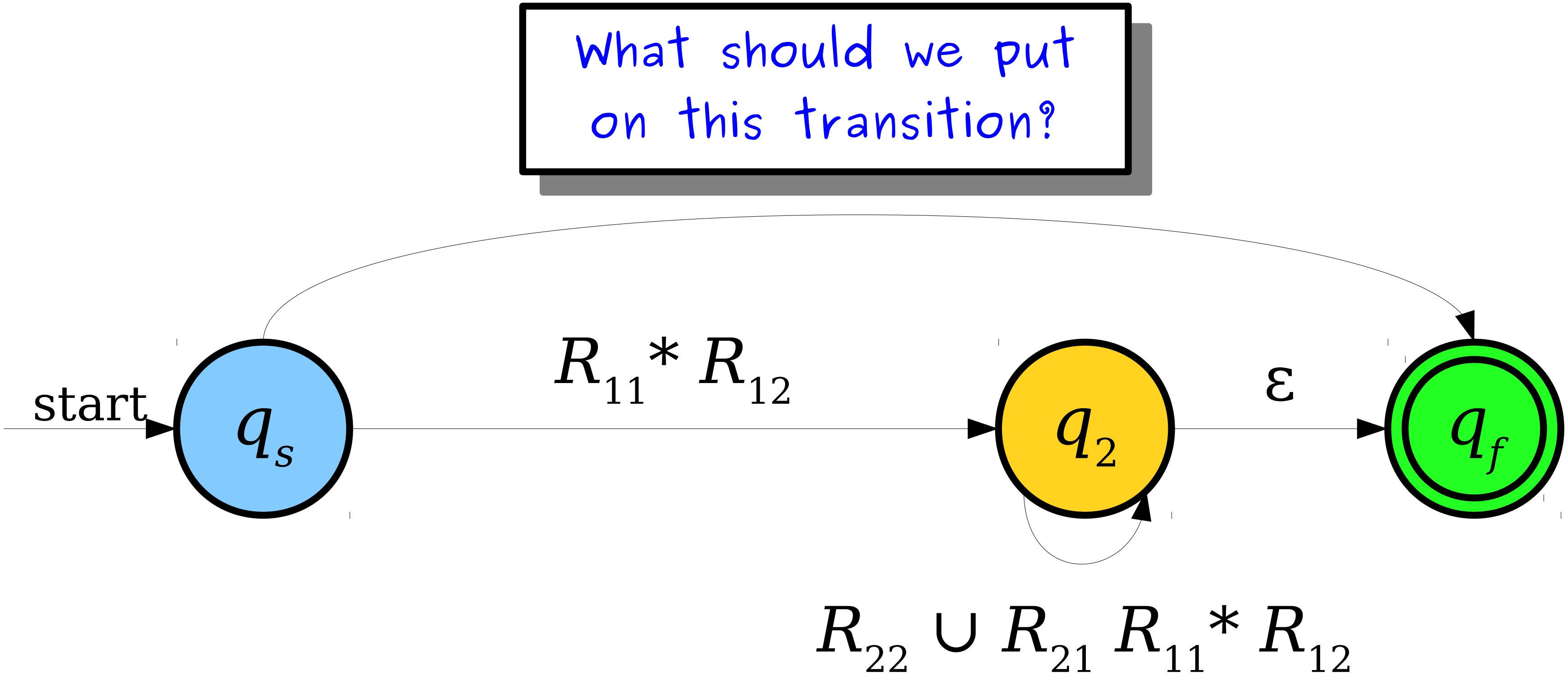

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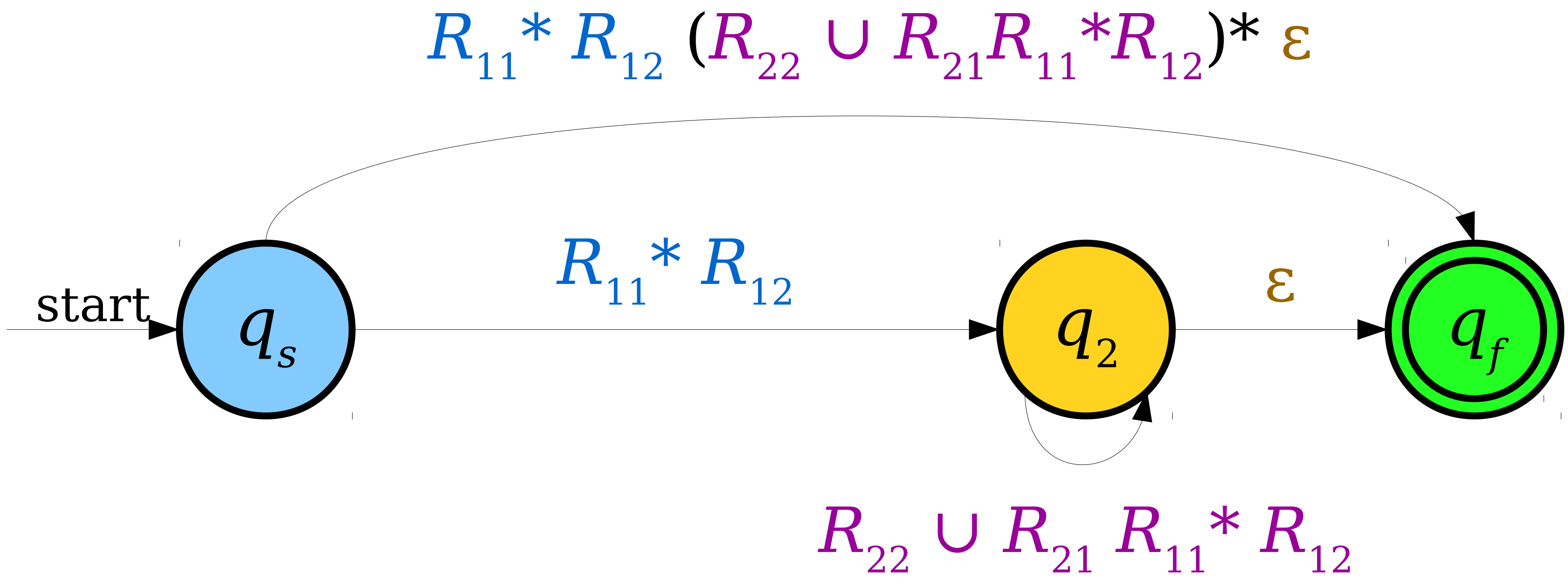

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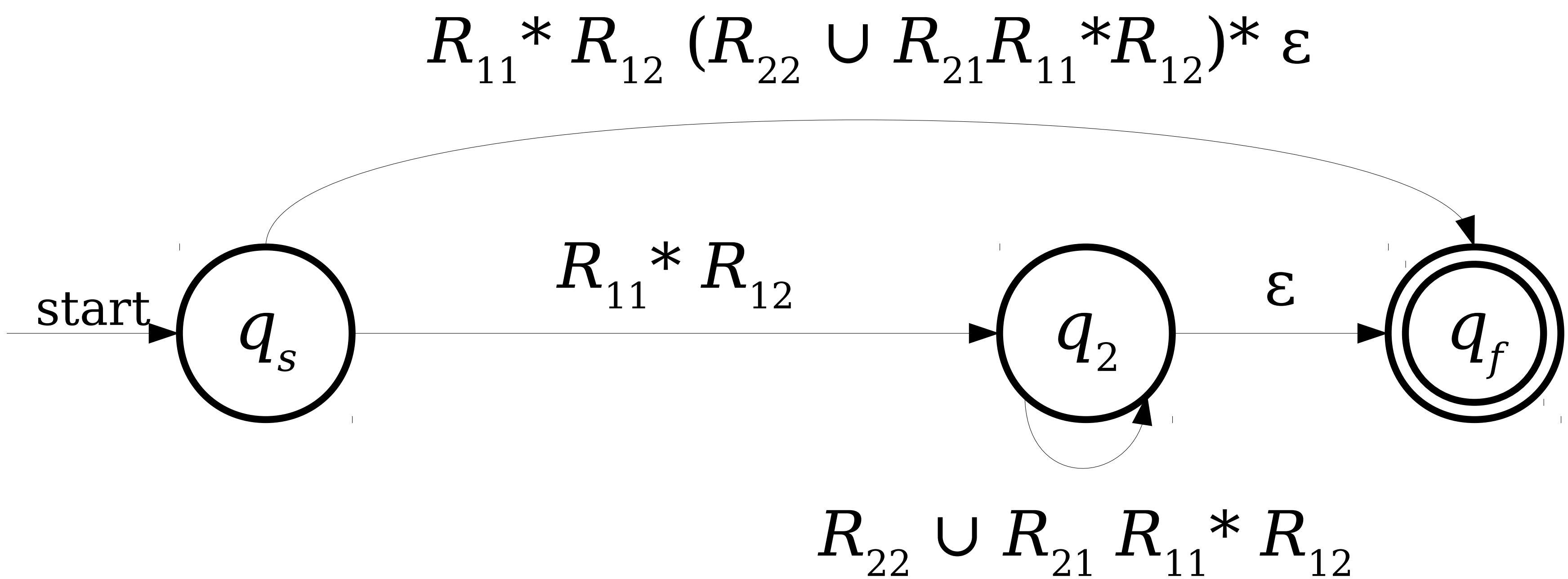

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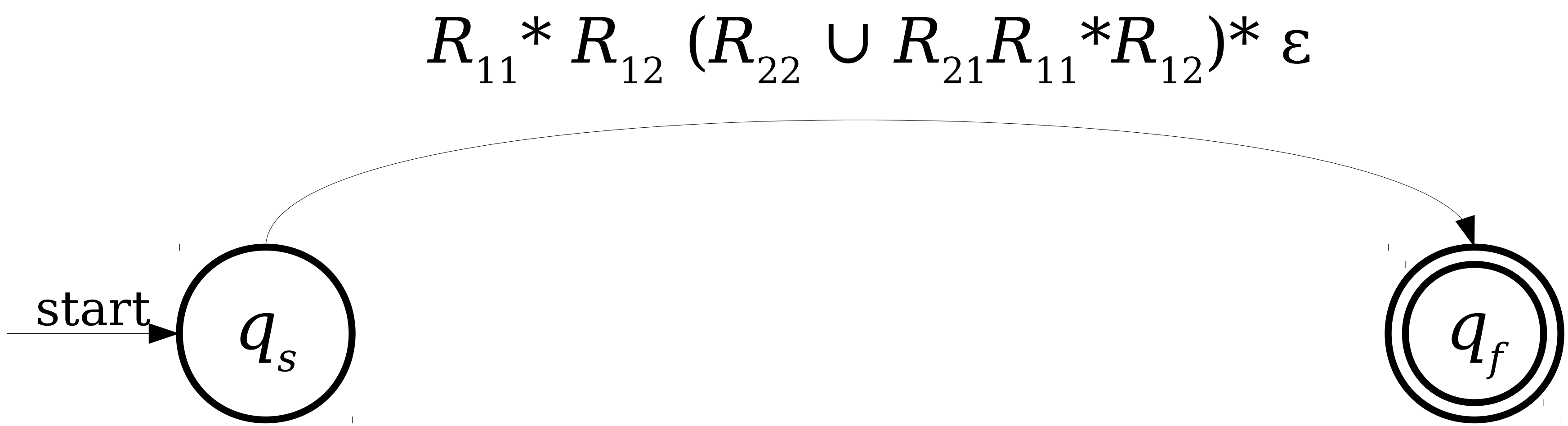

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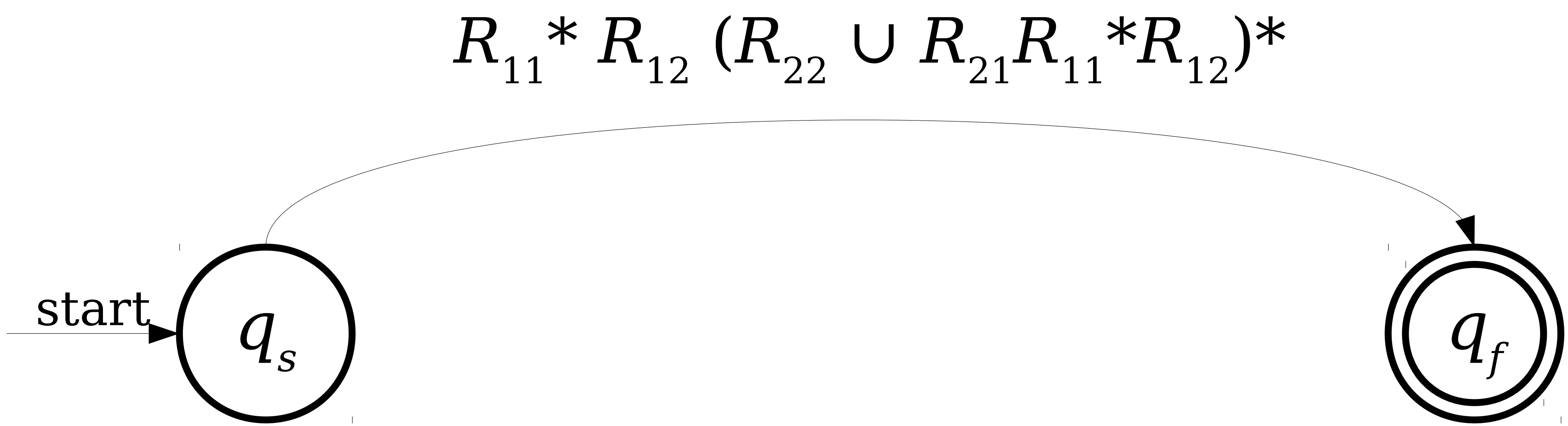

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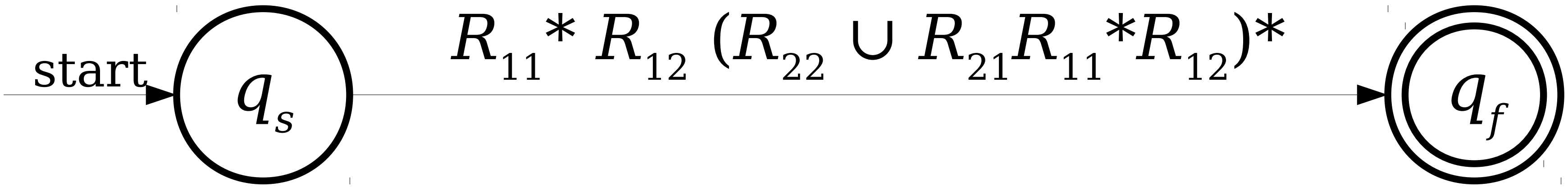

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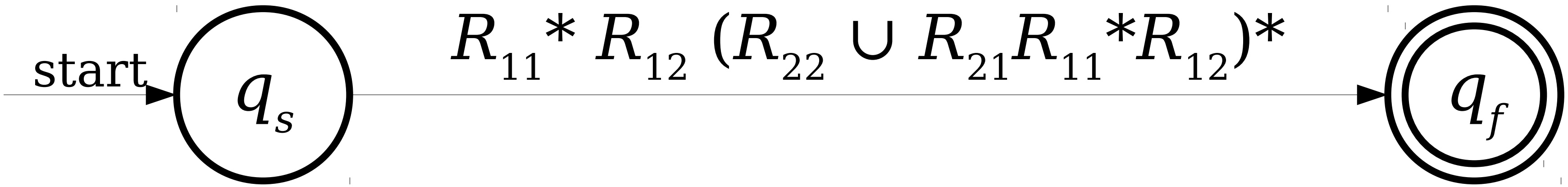
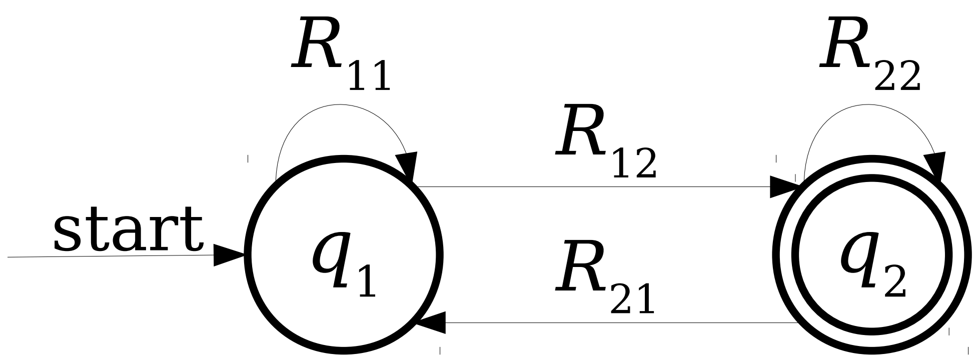




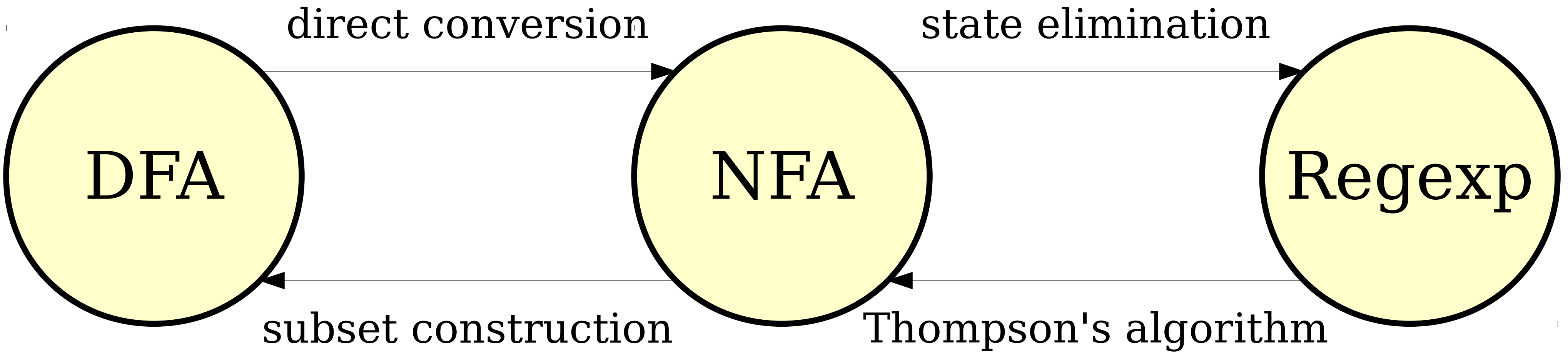
The Construction at a Glance

- Start with an NFA N for the language L.
- Add a new start state q_s and accept state q_f to the NFA.
  - Add an ε-transition from q_s to the old start state of N.
  - Add ε-transitions from each accepting state of N to q_f, then mark them as not accepting.
- Repeatedly remove states other than q_s and q_f from the NFA by "shortcutting" them until only two states remain: q_s and q_f.
- The transition from q_s to q_f is then a regular expression for the NFA.

Eliminating a State

- To eliminate a state q from the automaton, do the following for each pair of states q_0 and q_1, where there's a transition from q_0 into q and a transition from q into q_1:
  - Let R_{in} be the regex on the transition from q_0 to q.
  - Let R_{out} be the regex on the transition from q to q_1.
  - If there is a regular expression R_{stay} on a transition from q to itself, add a new transition from q_0 to q_1 labeled $((R_{in})(R_{stay})^*(R_{out}))$.
  - If there isn't, add a new transition from q_0 to q_1 labeled $((R_{in})(R_{out}))$
- If a pair of states has multiple transitions between them labeled R_1, R_2, …, R_k, replace them with a single transition labeled $R_1 \cup R_2 \cup \ldots \cup R_k$.

Our Transformations



Theorem: The following are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathcal{L}(D) = L$.
- There is an NFA N such that $\mathcal{L}(N) = L$.
- There is a regular expression R such that $\mathcal{L}(R) = L$.

Why This Matters

- The equivalence of regular expressions and finite automata has practical relevance.
  - Tools like `grep` and `flex` that use regular expressions capture all the power available via DFAs and NFAs.
- This also is hugely theoretically significant: the regular languages can be assembled “from scratch” using a small number of operations!

Let’s take a five minute break!

Art by 白砂糖
OREO
O&REO
O&O
OREOREO
RERERERERE
OOOOO
OREOO
OREORERERERE ORE
OREOREORE
REREO
REORE
ORERERERERERERERERO
OOORERERERERERERERE OOO
ORERERERE OOOOOOOOO
Typeset by Ree
Translated by Mysterious Uploader (danbooru)

Oreo Sandwiches

- Let Σ = { **O**, **R** }

For simplicity, let's just use a single character for the "cream" part of the Oreo :)

Oreo Sandwiches

- Let $\Sigma = \{$ **O**, **R** $\}$

Design a DFA for the language

$$L = \{\ w \in \Sigma^* \mid w \neq \varepsilon \text{ and the first and last character of } w \text{ are the same }\}$$

Oreo Sandwiches

- Let $\Sigma = \{$ **O**, **R** $\}$

Design a DFA for the language

$$L = \{ w \in \Sigma^* \mid w \neq \varepsilon \text{ and the first and last character of } w \text{ are the same } \}$$

- **ORO** $\in L$
- **ROOOR** $\in L$
- **OROORORRO** $\in L$
- **OR** $\notin L$
- **OOOOOR** $\notin L$
- **ROROROR0** $\notin L$

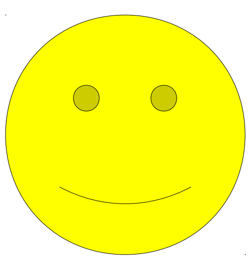
Designing DFAs

- ***States*** – pieces of information
    - What do I have to keep track of in the course of figuring out whether a string is in this language?
- ***Transitions*** – updating state
    - From the state I'm currently in, what do I know about my string? How would reading this character change what I know?

An Analogy

Imagine a scenario where Bob is thinking of a string and Alice has to figure out whether that string is in a particular language

$L = \{ w \text{ is divisible by } 5 \}$



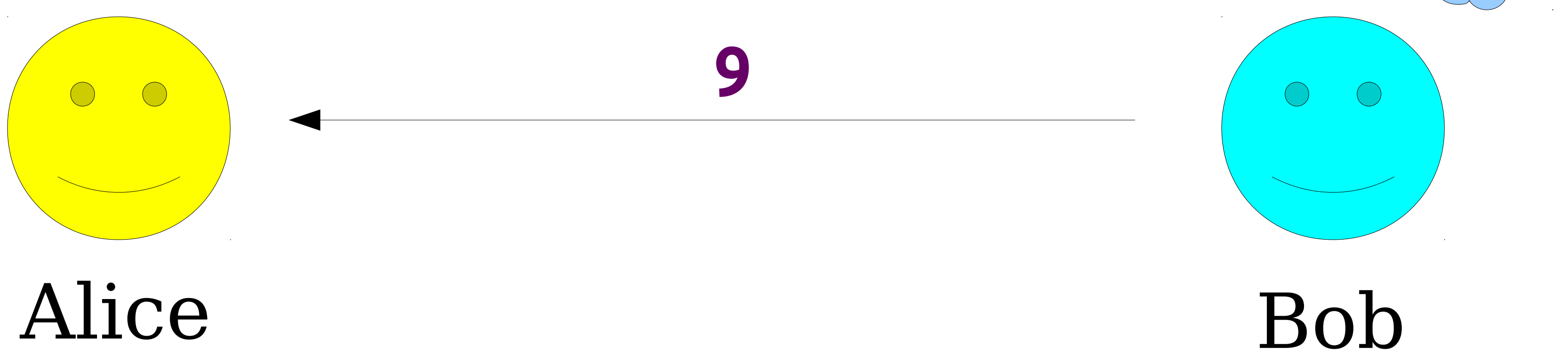
Alice

Bob

An Analogy

The catch: Bob can only send Alice one character at a time, and Alice doesn't know how long the string is until Bob tells her that he's done sending input

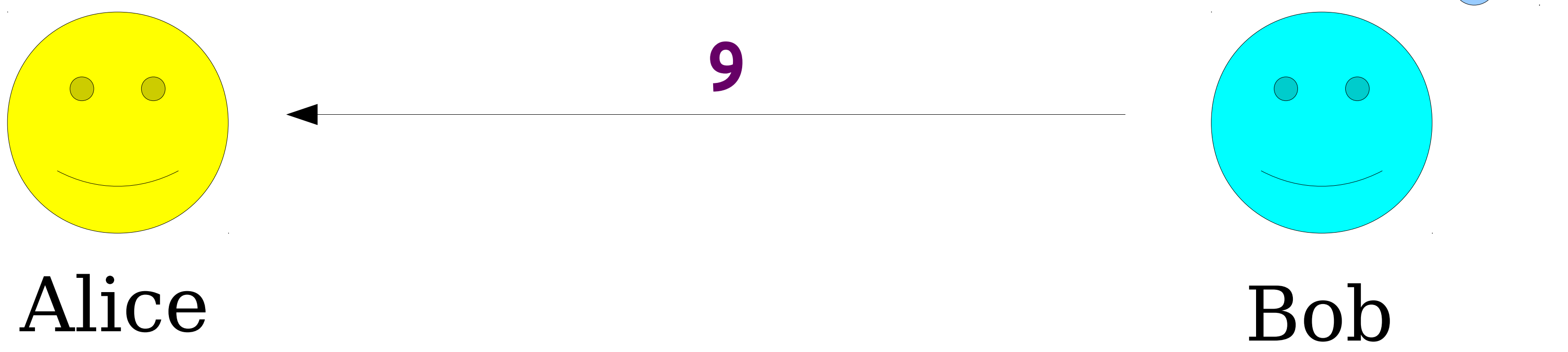
$L = \{ w \text{ is divisible by 5} \}$



An Analogy

What does Alice need to remember about the characters she's receiving from Bob?

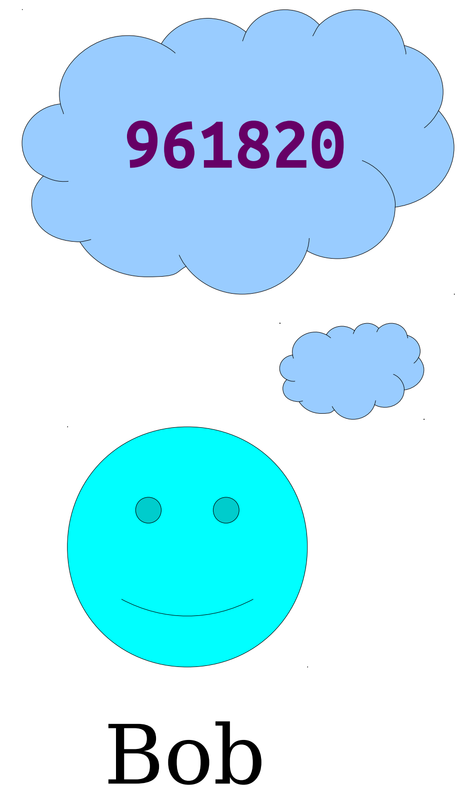
$L = \{ w \text{ is divisible by 5 } \}$



An Analogy

Key insight: Alice only needs to remember the last character she received from Bob

$L = \{ w \text{ is divisible by } 5 \}$



An Analogy

Key insight: Alice only needs to remember the last character she received from Bob

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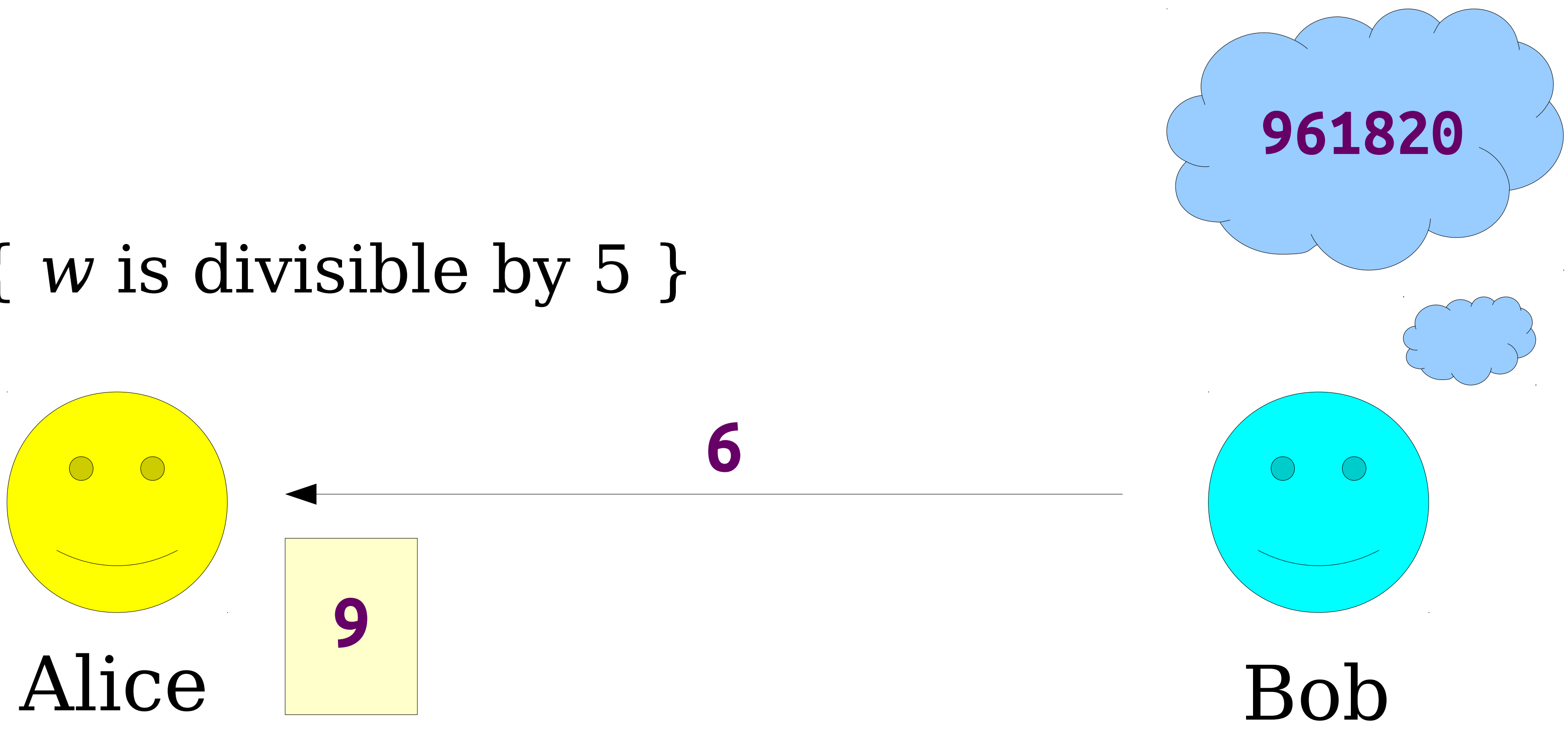
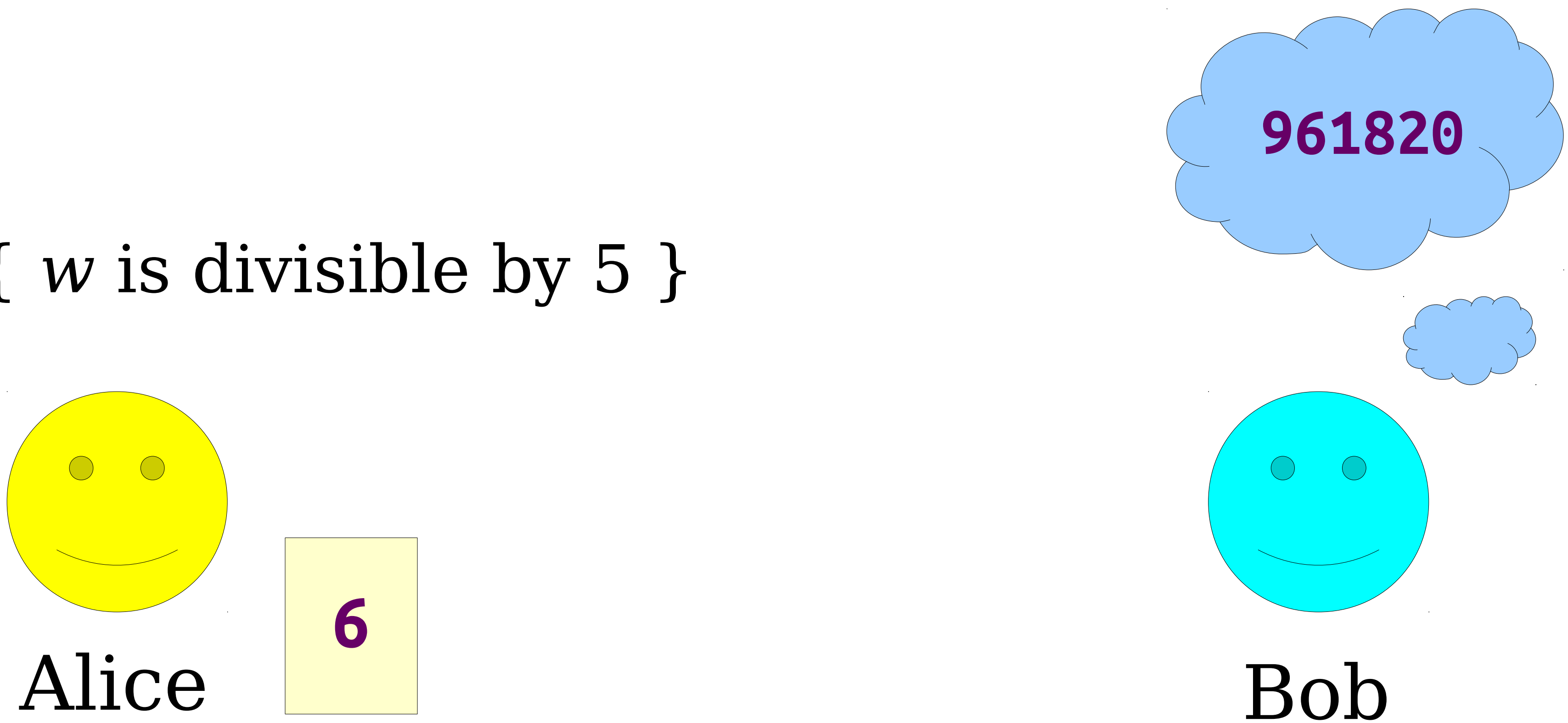


An Analogy

Key insight: Alice only needs to remember the last character she received from Bob

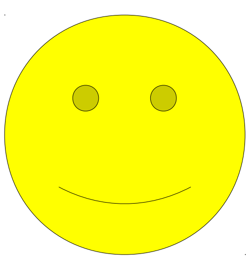
$L = \{ w \text{ is divisible by } 5 \}$



An Analogy

Key insight: Alice only needs to remember the last character she received from Bob

$L = \{\ w \text{ is divisible by } 5\ \}$



Alice

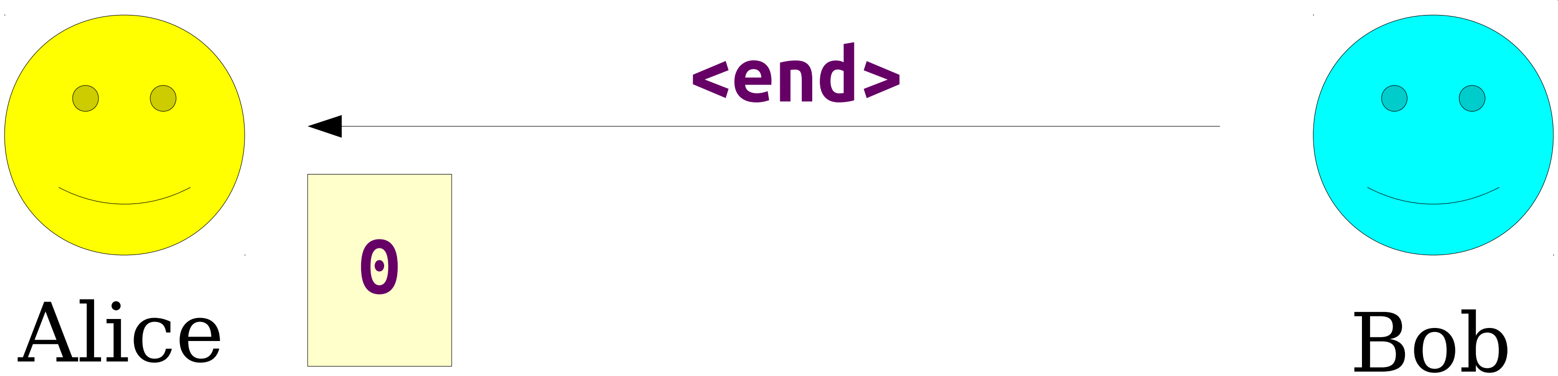
. . .

Bob

An Analogy

Eventually Bob gets to the end of his string and sends Alice a signal that he’s done sending input

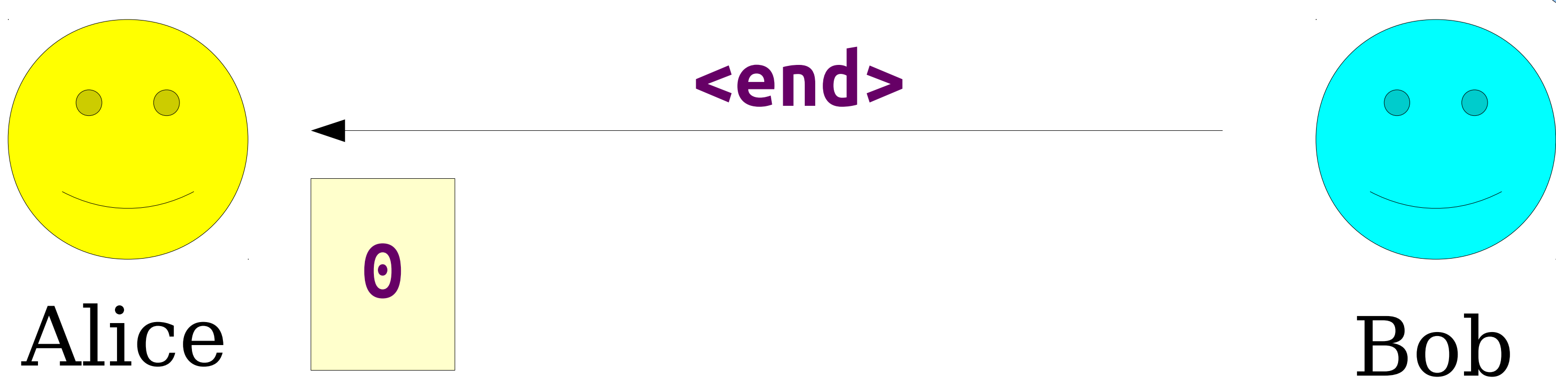
$L = \{\ w \text{ is divisible by } 5\ \}$



An Analogy

At this point, Alice just has to look at the last digit she wrote down and if it’s a 5 or 0, Bob’s string belongs in the language

$L = \{ w \text{ is divisible by 5 } \}$



DFA Design Strategy

1. Answer the question "What do I have to keep track of in the course of figuring out whether a string is in this language?"

2. Create a state that represents each possible answer to that question.

3. From each state, go through all of the characters and answer the question "How would reading this character change what I know about my string?" and draw transitions to the appropriate states.

DFA Design Strategy

$L = \{\ w \text{ is divisible by } 5\ \}$

1. Answer the question "What do I have to keep track of in the course of figuring out whether a string is in this language?"

We need to keep track of the last character.

2. Create a state that represents each possible answer to that question.

The last character could be any digit 0–9. The states for 0 and 5 are accepting states.

3. From each state, go through all of the characters and answer the question "How would reading this character change what I know about my string?" and draw transitions to the appropriate states.

Reading a character *d* should transition to the state representing "the last character of the string is *d*".

Oreo Sandwiches

- Let $\Sigma = \{$ **O**, **R** $\}$

Design a DFA for the language

$$L = \{ w \in \Sigma^* \mid w \neq \varepsilon \text{ and the first and last character of } w \text{ are the same } \}$$

What do I have to keep track of in the course of figuring out whether a string is in this language?

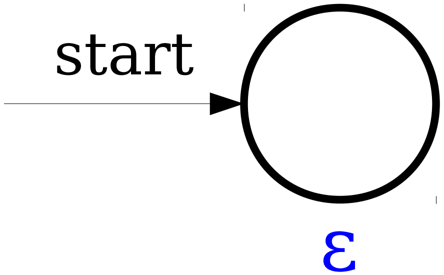
Oreo Sandwiches

$$L = \{\ w \in \Sigma^* \mid w \neq \varepsilon \text{ and the first and last character of } w \text{ are the same }\}$$

- We need to keep track of the very first character
- And we need to keep track of the last character we've read so that when we reach the end, we can check whether the first and last characters were the same

Oreo Sandwiches

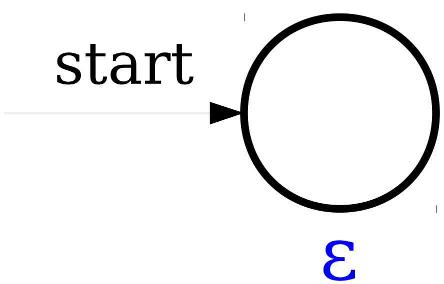
$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



Remember that each state should represent a piece of information. We'll annotate what each state represents in blue.

Oreo Sandwiches

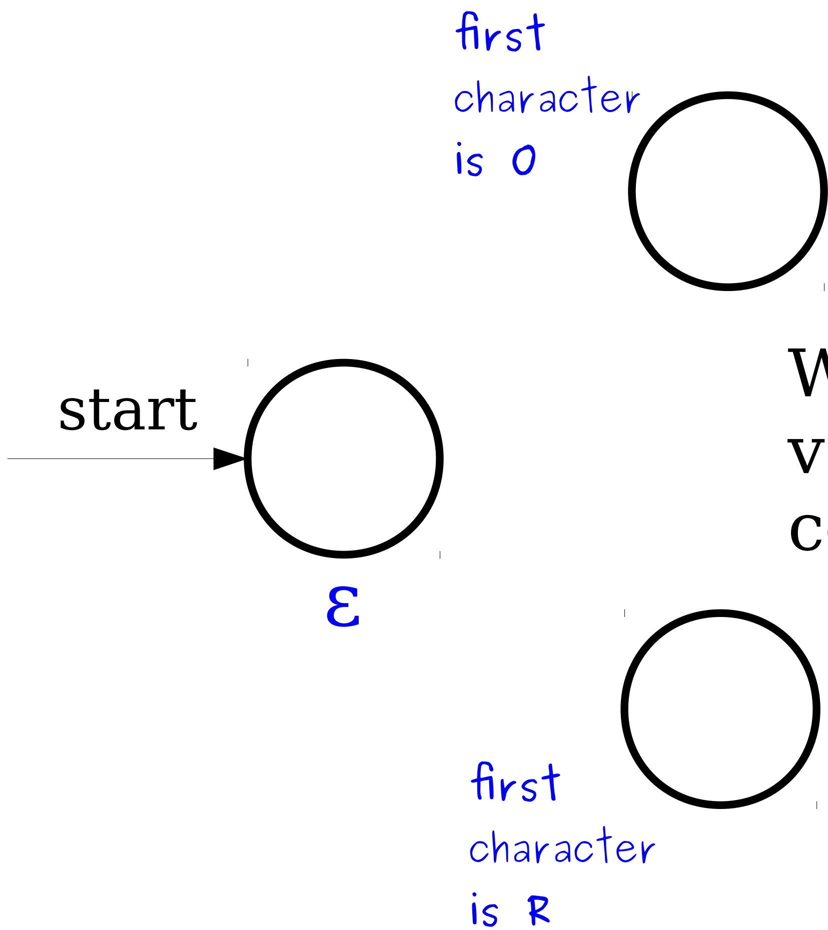
$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



We need to keep track of the very first character, which could either be an **O** or an **R**

Oreo Sandwiches

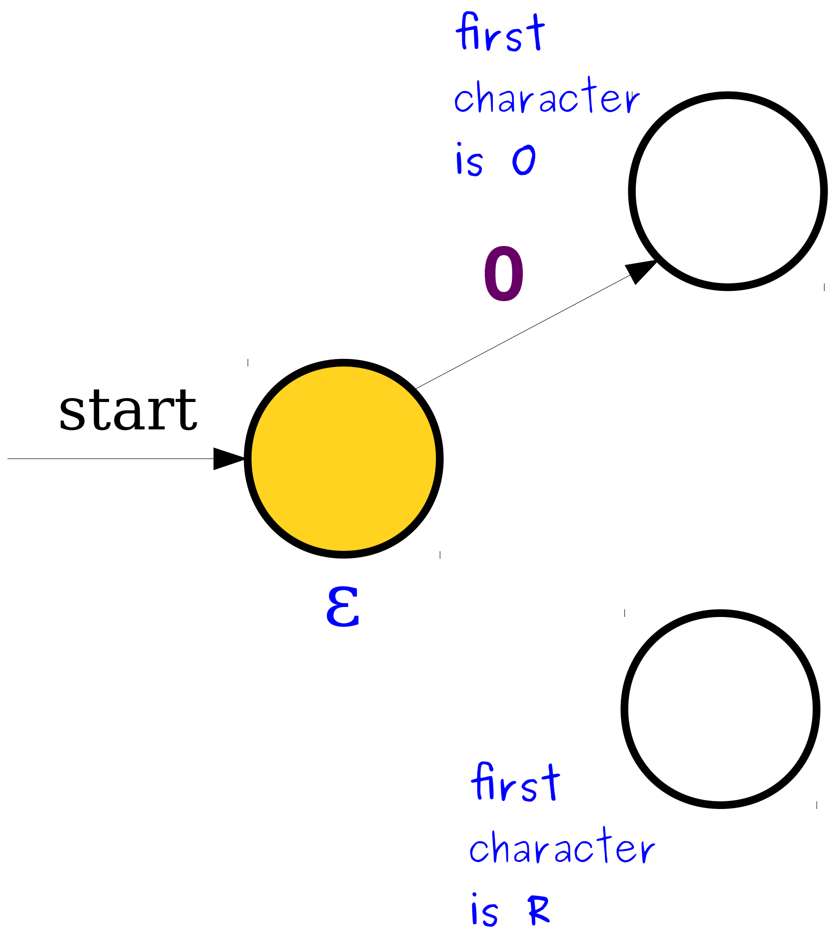
$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



We need to keep track of the very first character, which could either be an **O** or an **R**

Oreo Sandwiches

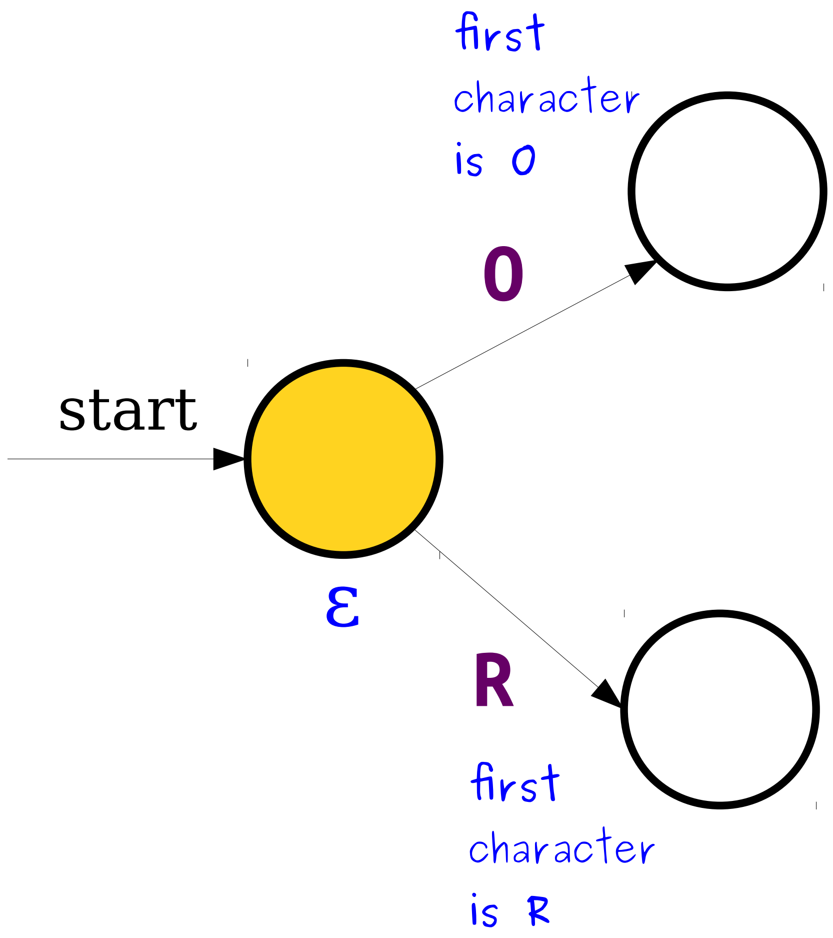
L = { $w \in \Sigma^*$ | $w \neq \varepsilon$ and the first and last character of w are the same }



If I'm in the start state and I read an **0**, I should transition to this state

Oreo Sandwiches

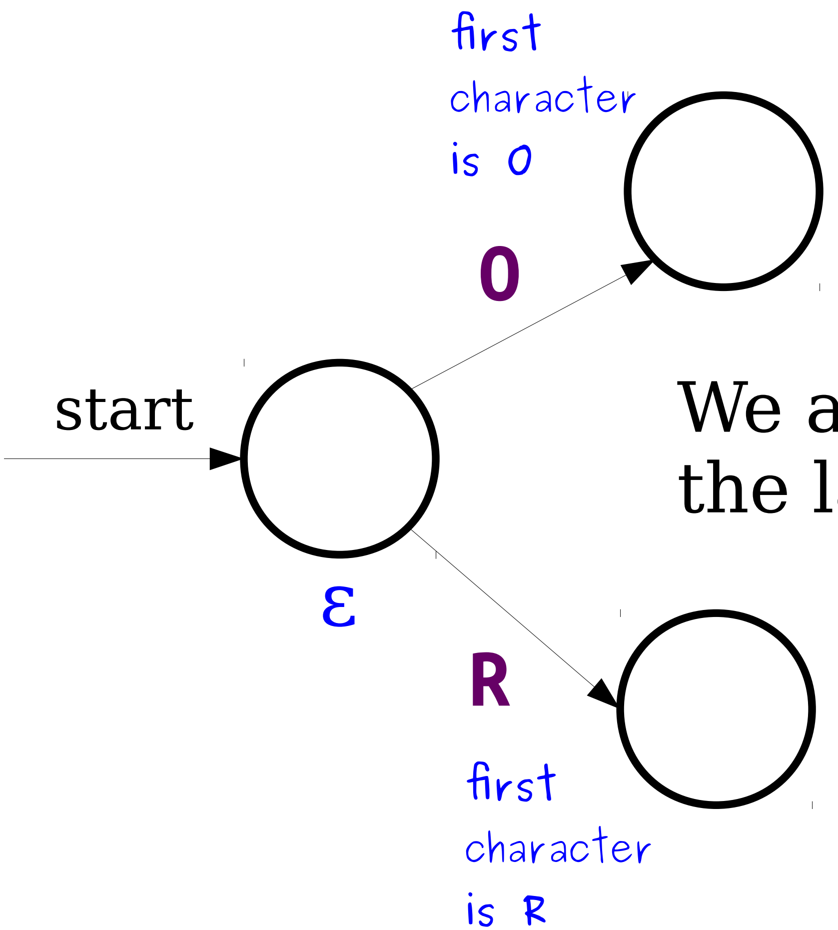
$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



Likewise if I'm in the start state and I read an **R**, I should transition to this state

Oreo Sandwiches

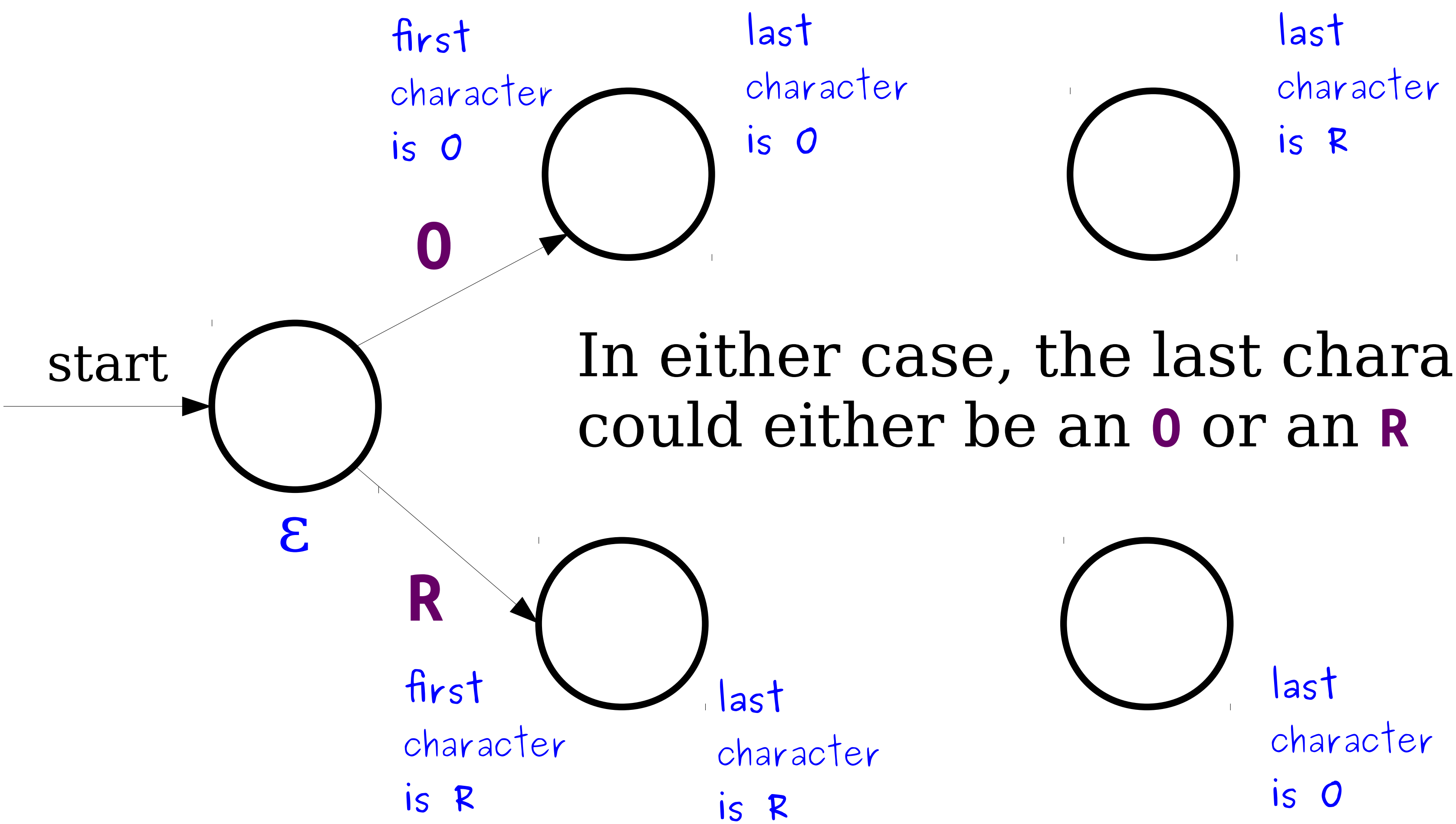
$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



We also need to keep track of the last character we've read

Oreo Sandwiches

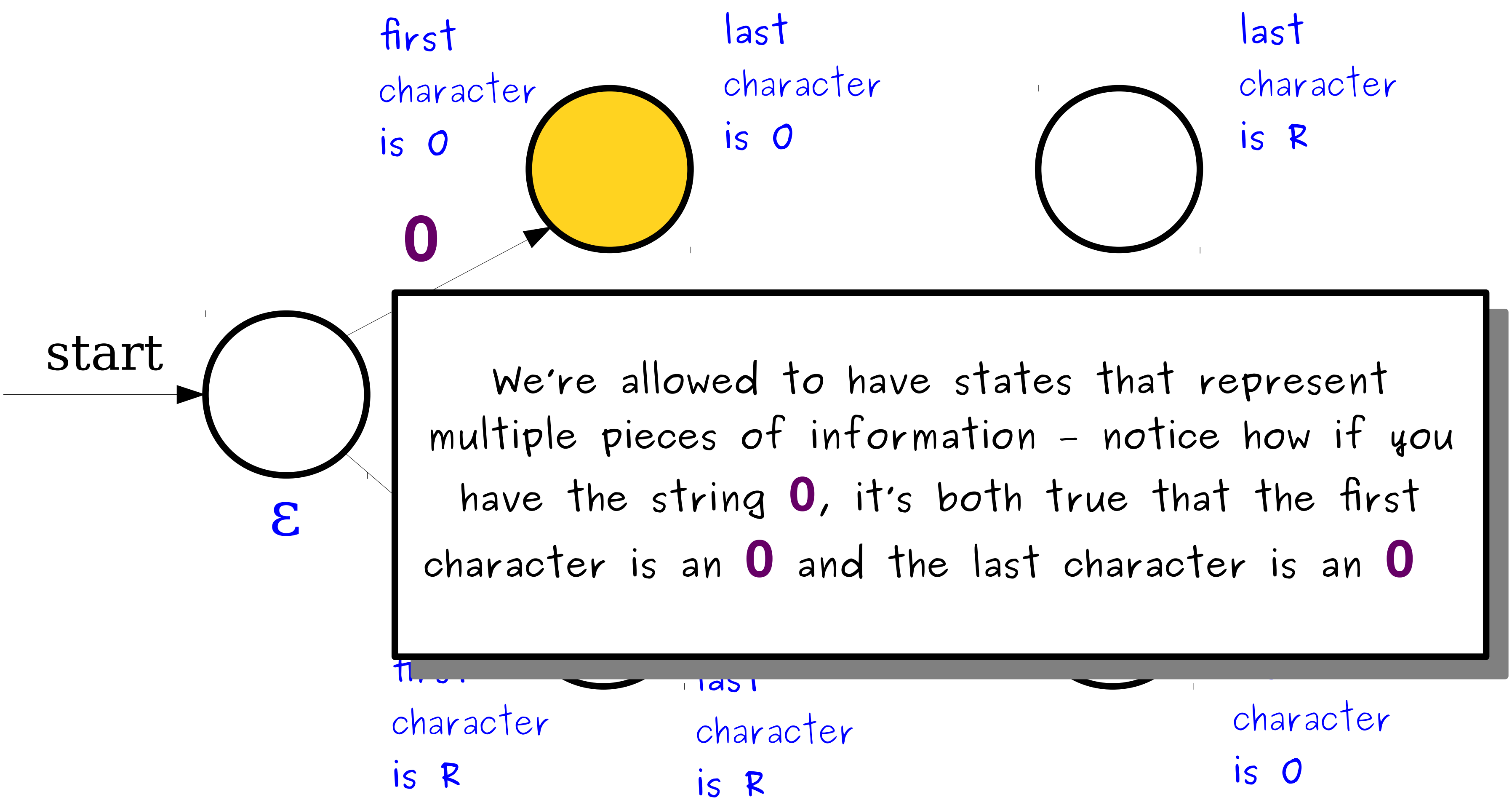
$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



In either case, the last character could either be an **O** or an **R**

Oreo Sandwiches

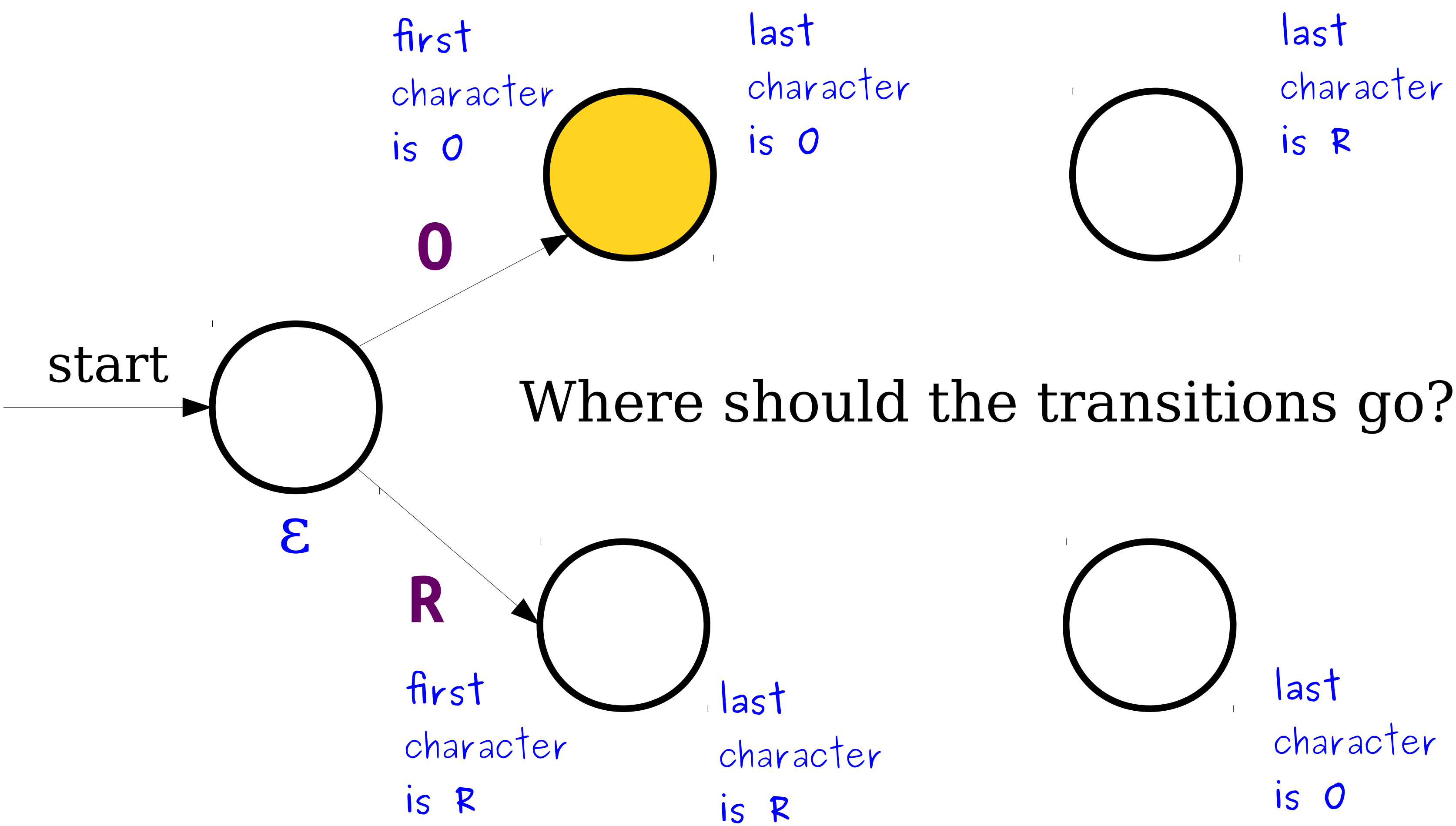
$$L = \{\ w \in \Sigma^* \mid w \neq \varepsilon \text{ and the first and last character of } w \text{ are the same }\}$$



We're allowed to have states that represent multiple pieces of information – notice how if you have the string **0**, it's both true that the first character is an **0** and the last character is an **0**

Oreo Sandwiches

$L = \{ \; w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



Oreo Sandwiches

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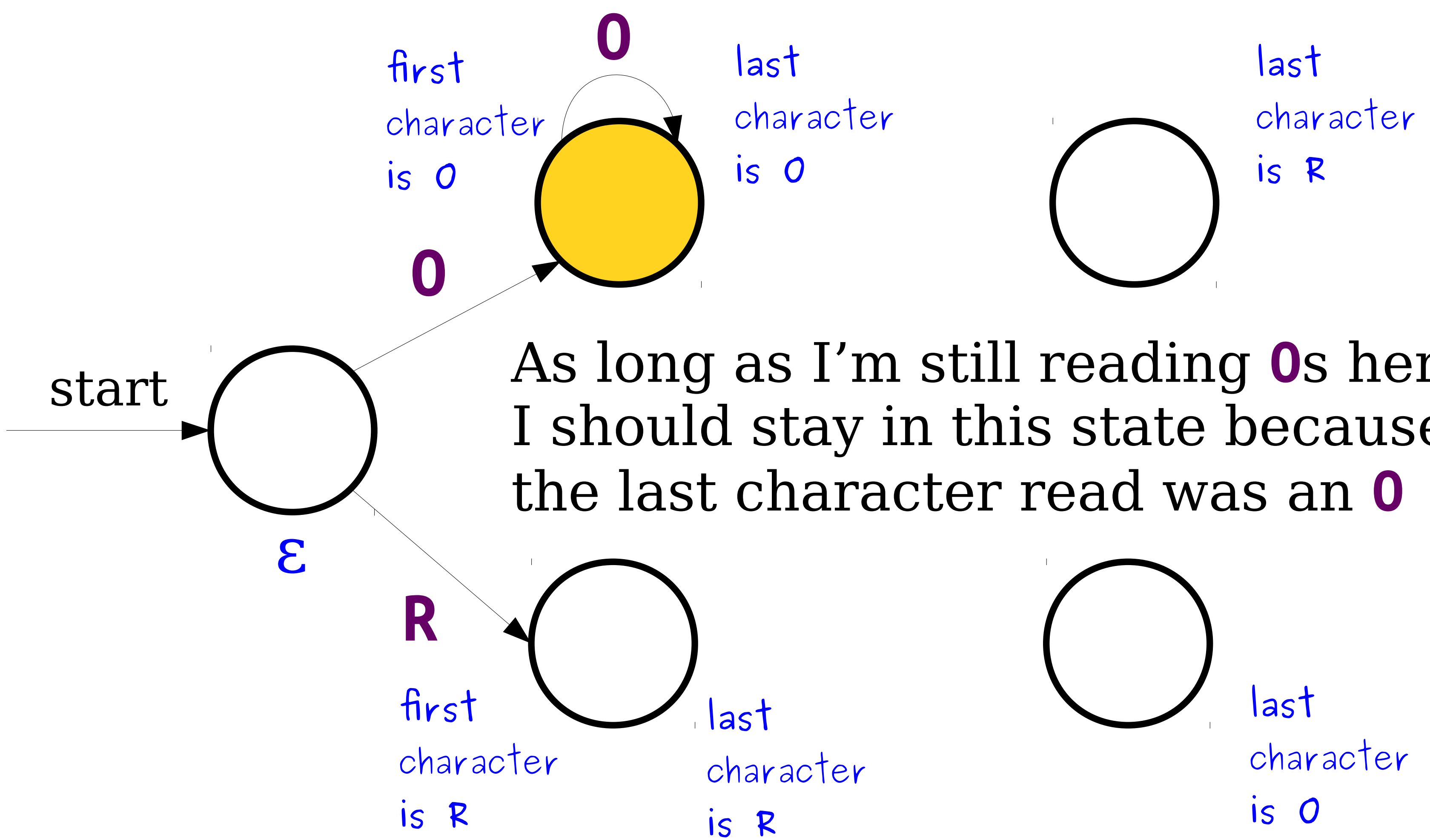
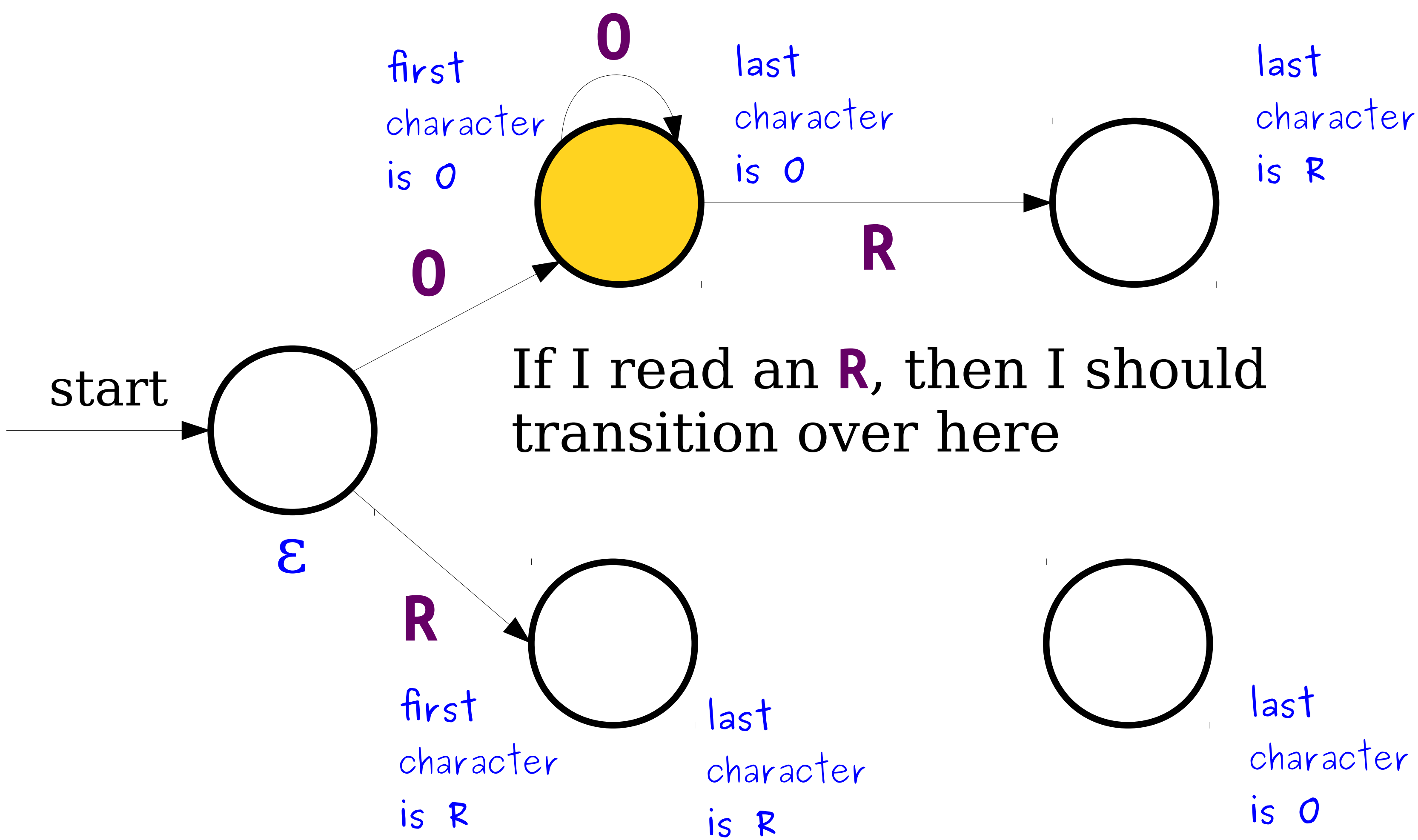


As long as I'm still reading **0**s here, I should stay in this state because the last character read was an **0**

Oreo Sandwiches

$L = \{\ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



Oreo Sandwiches

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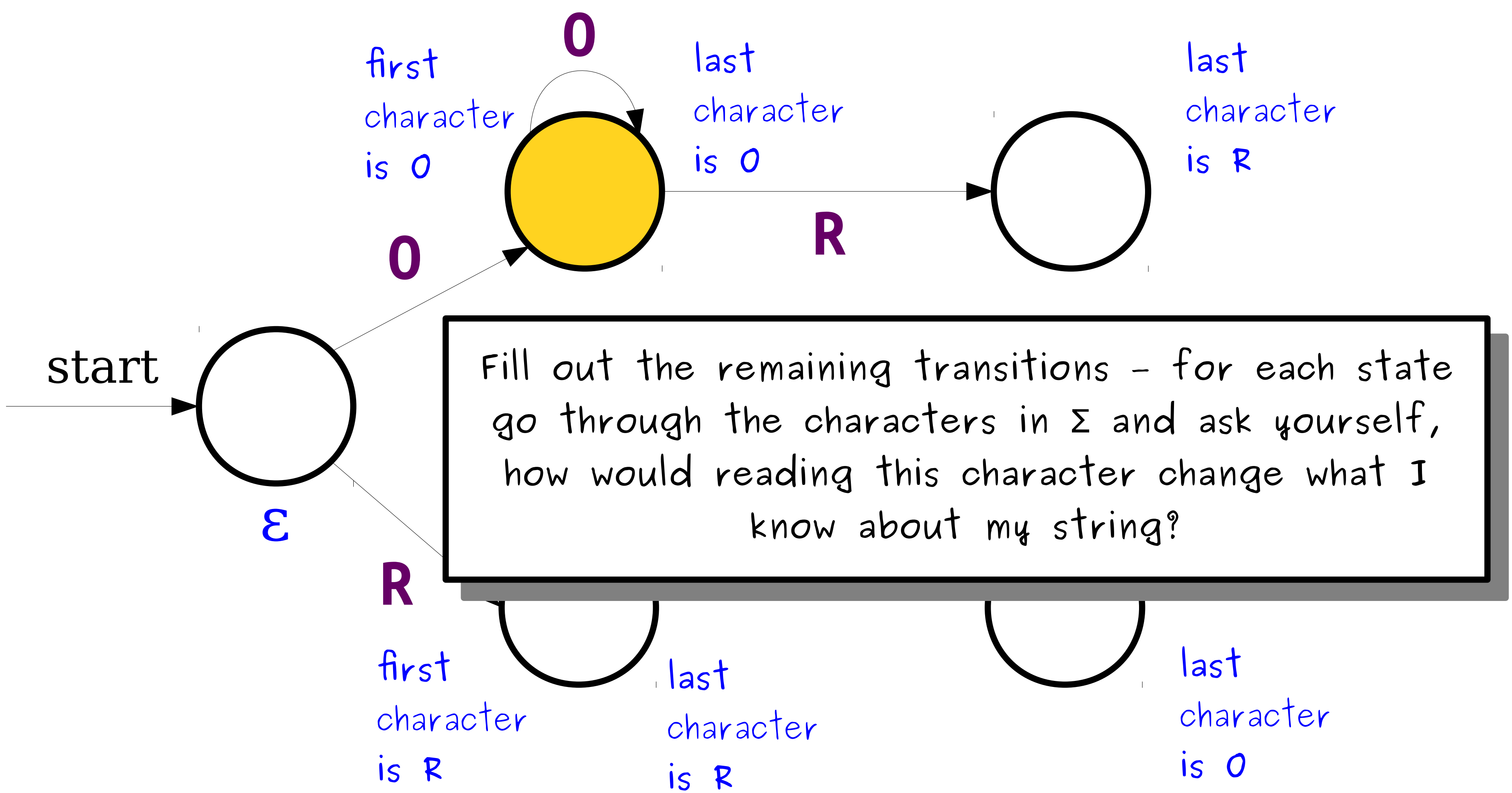

Oreo Sandwiches

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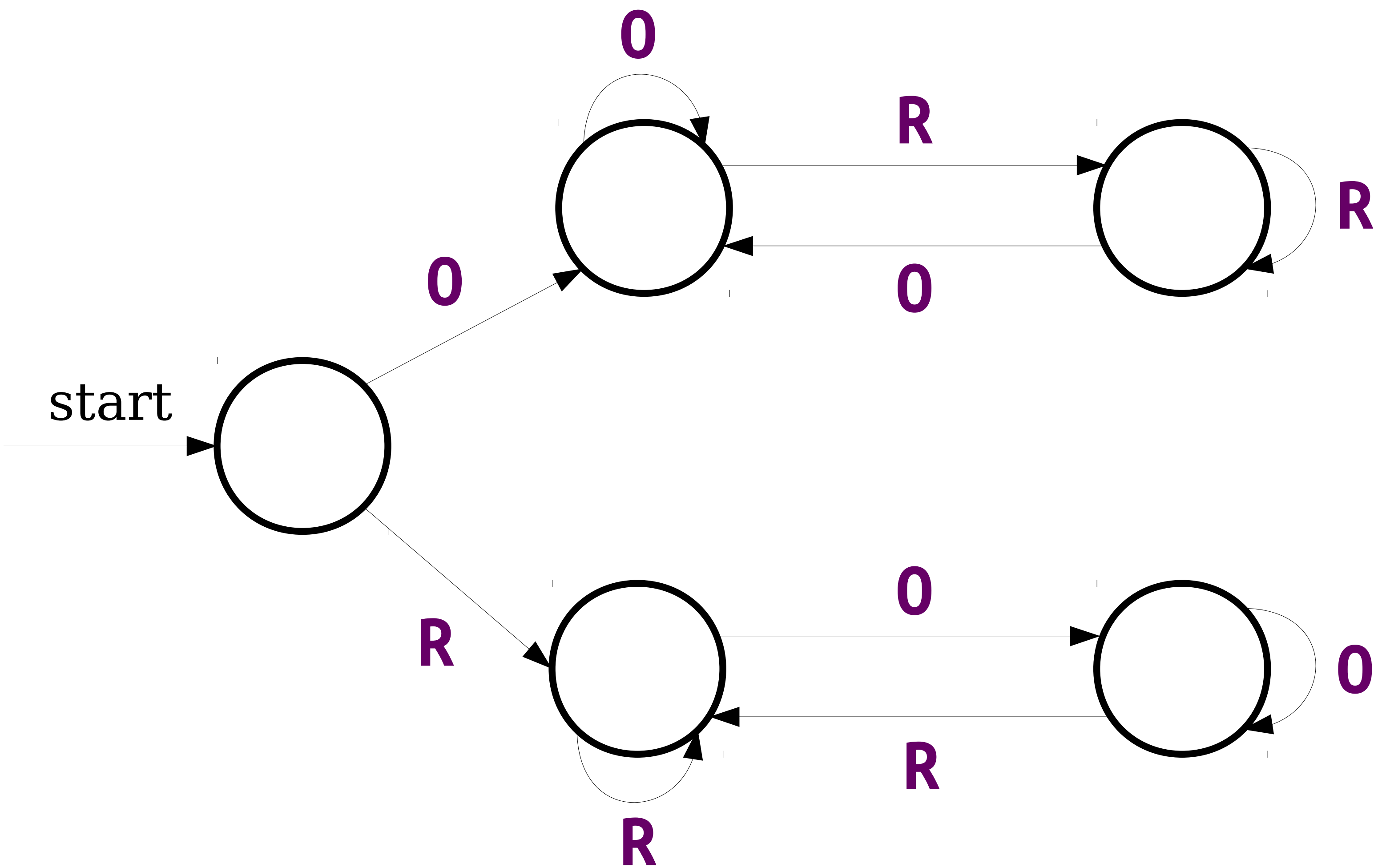

Oreo Sandwiches

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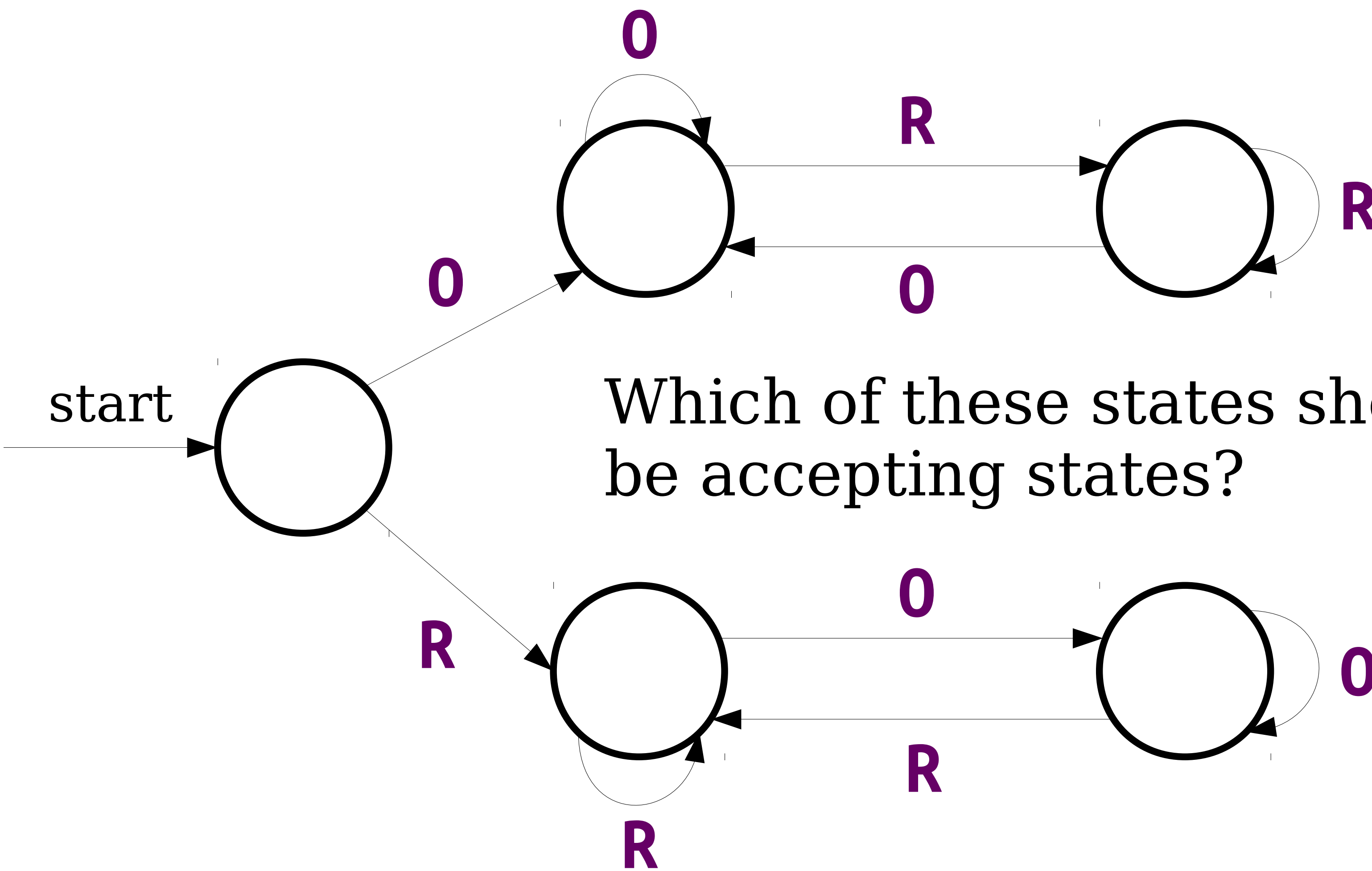
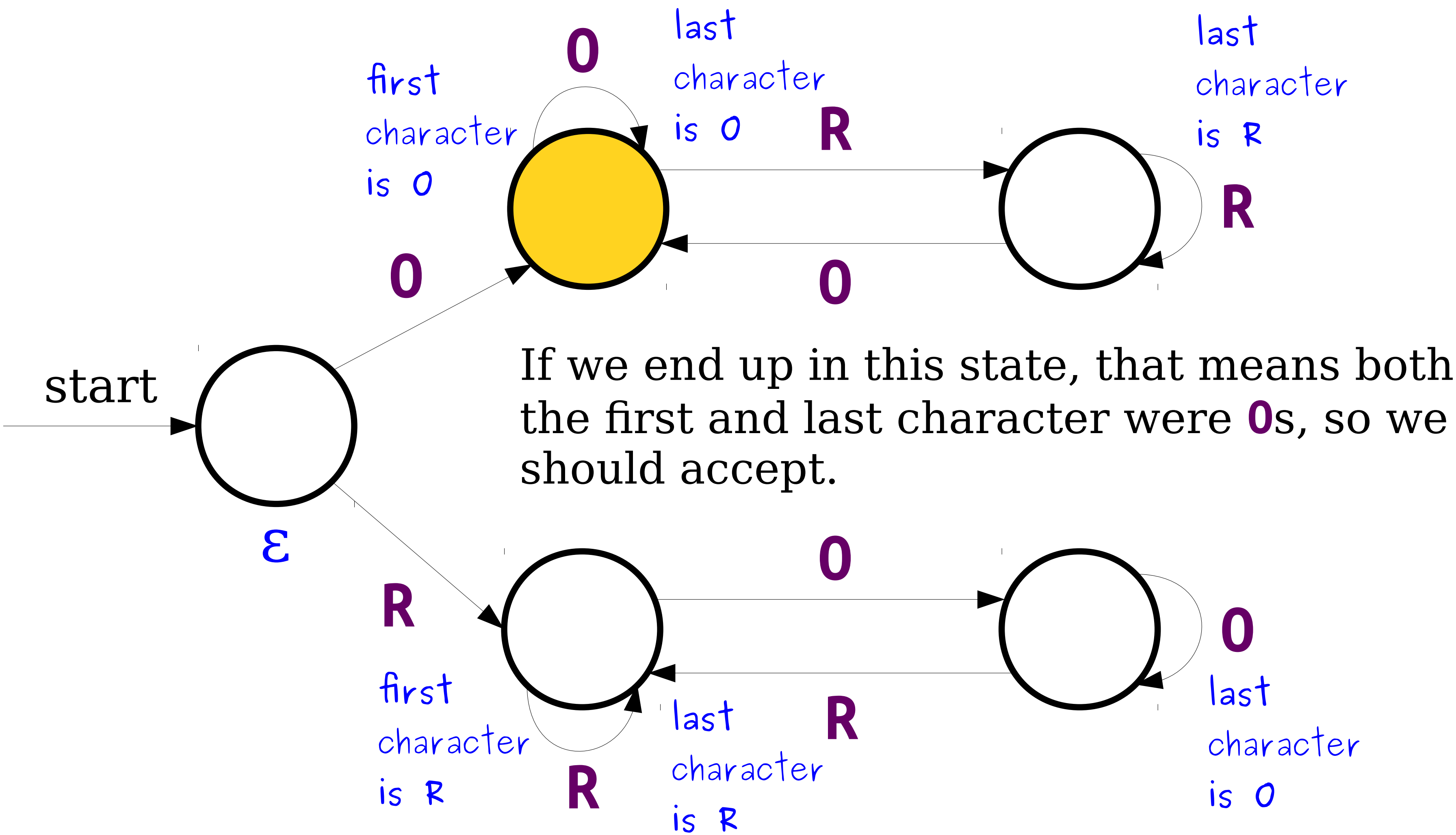


Which of these states should be accepting states?

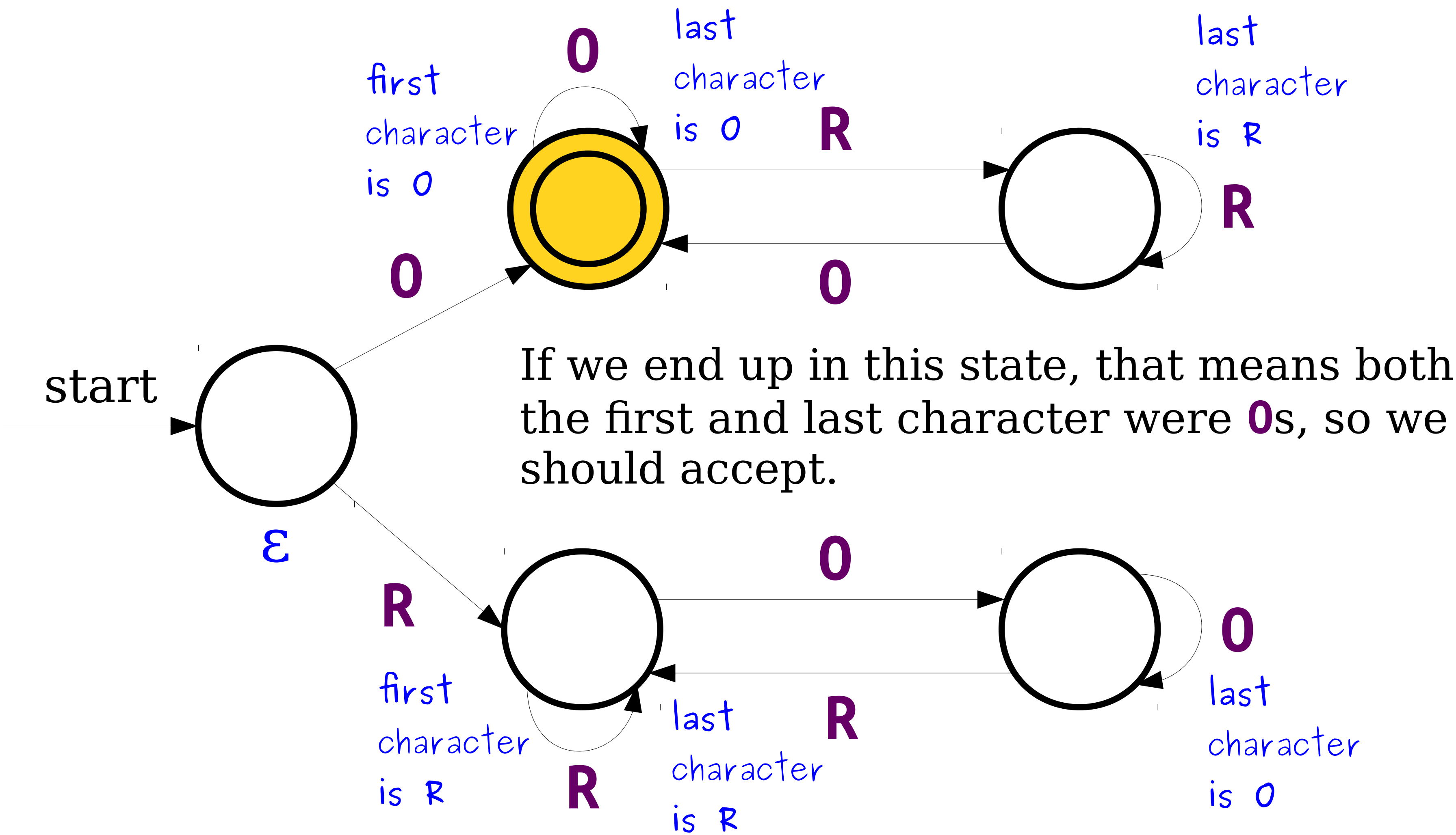
Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



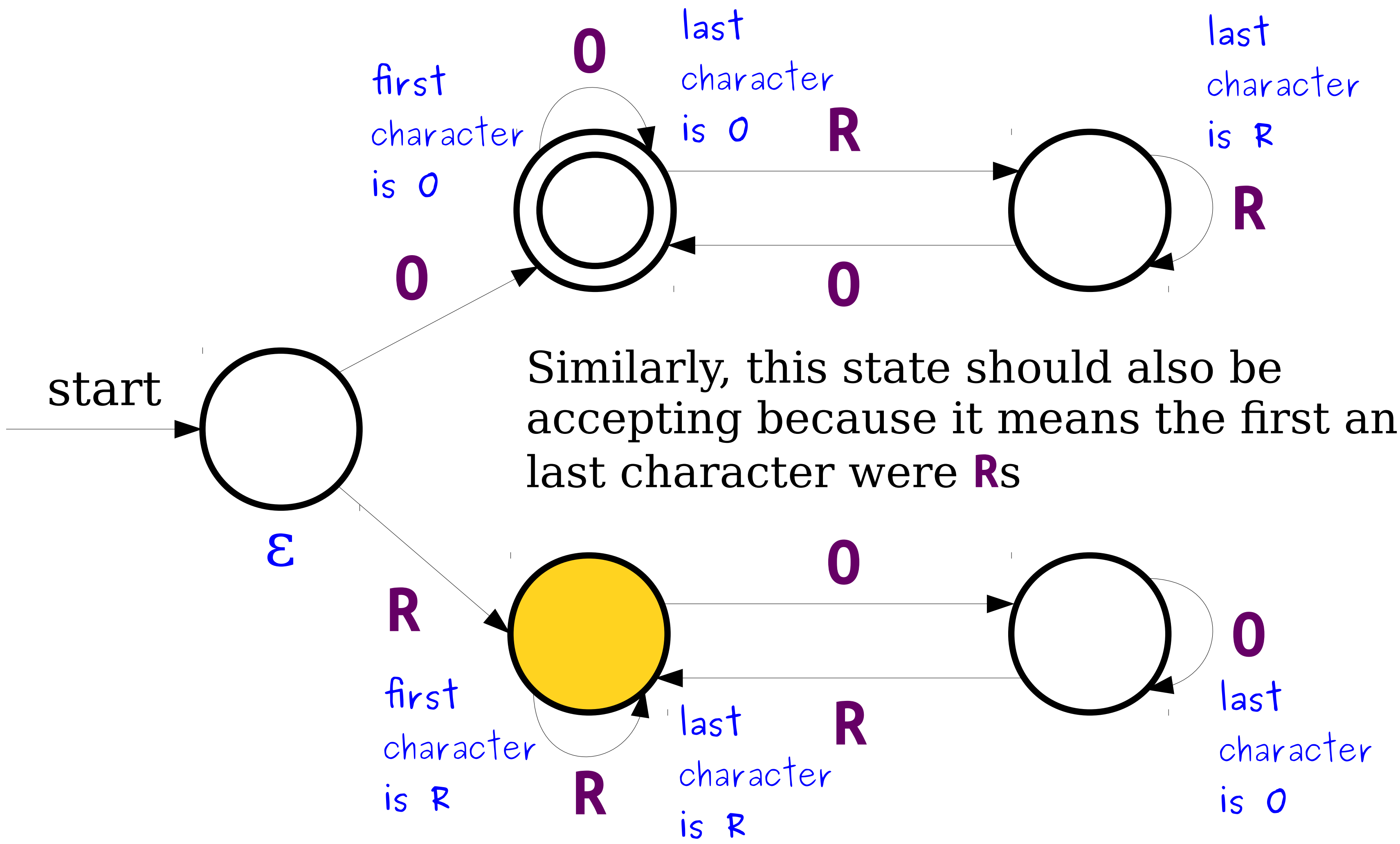
Oreo Sandwiches

$L = \{ \, w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



Oreo Sandwiches

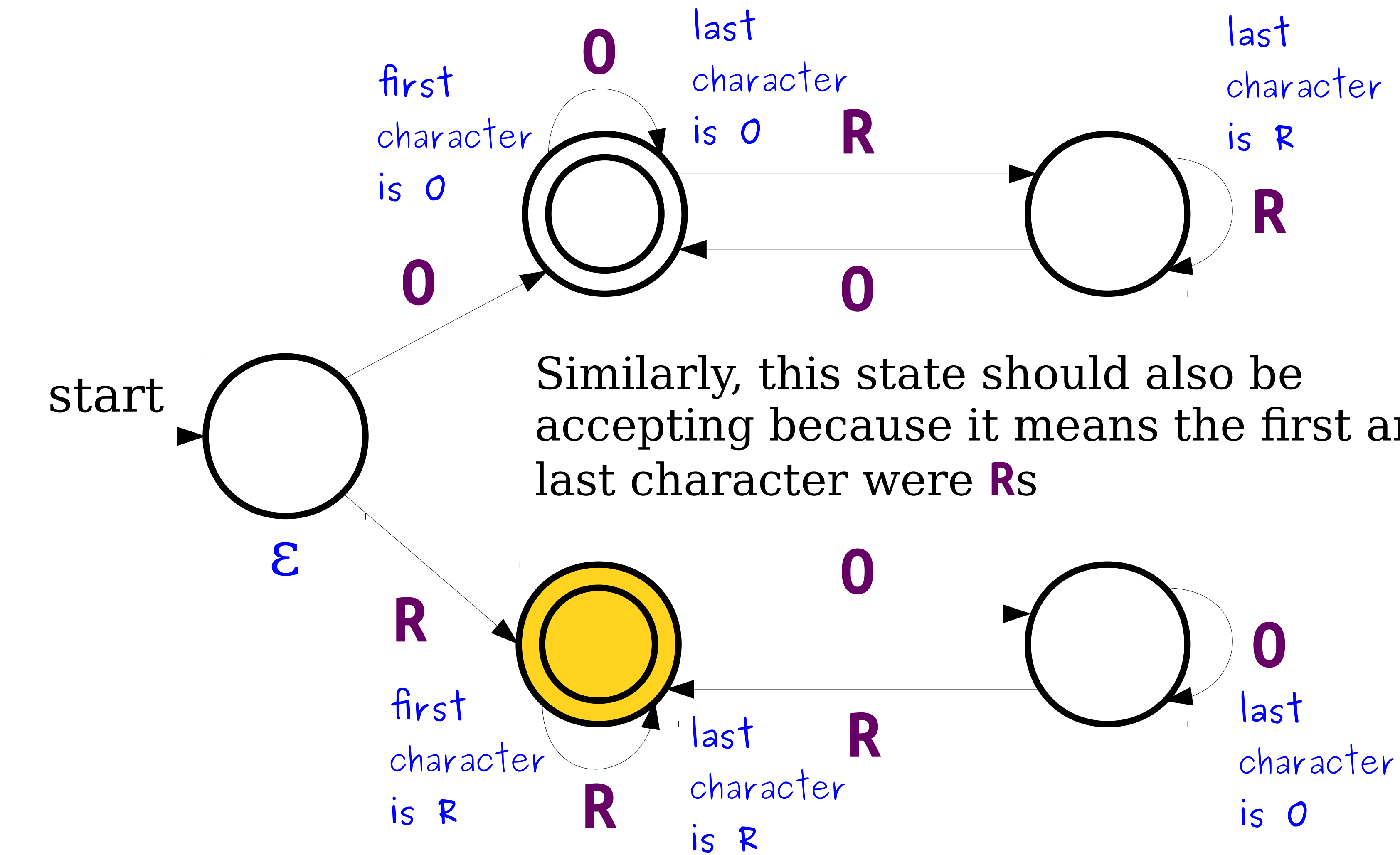
$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



Similarly, this state should also be accepting because it means the first and last character were **R**s

Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



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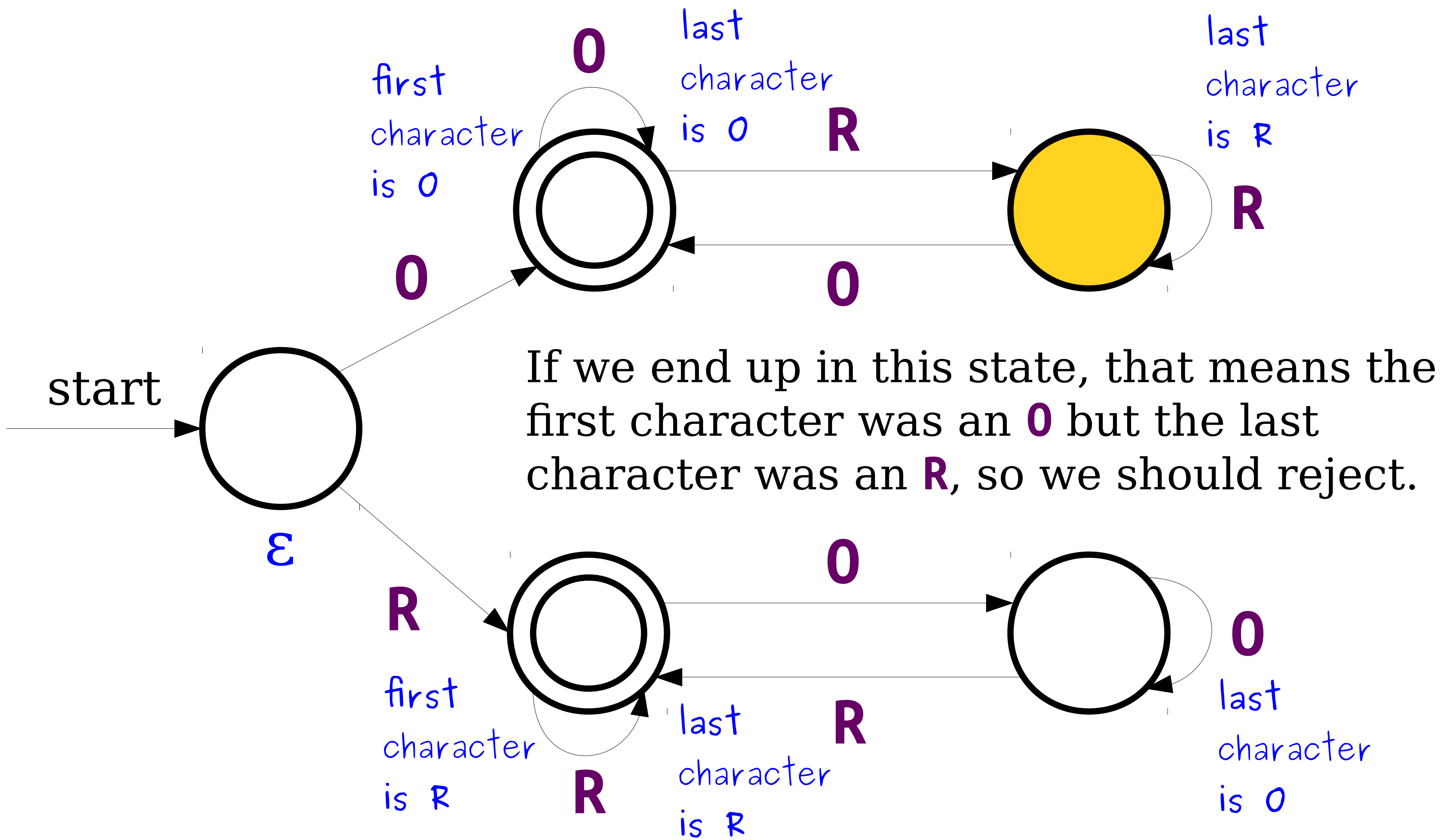
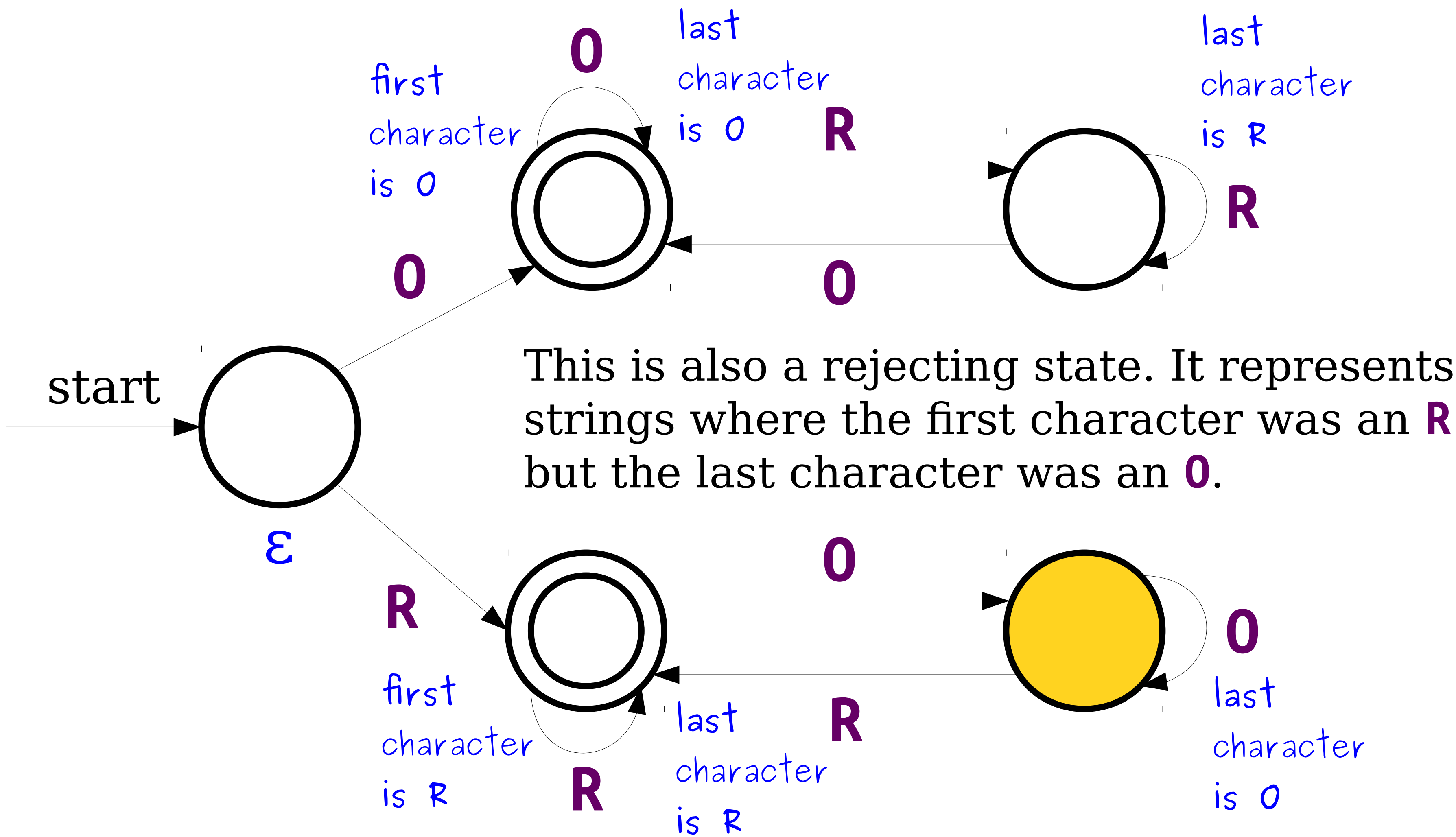


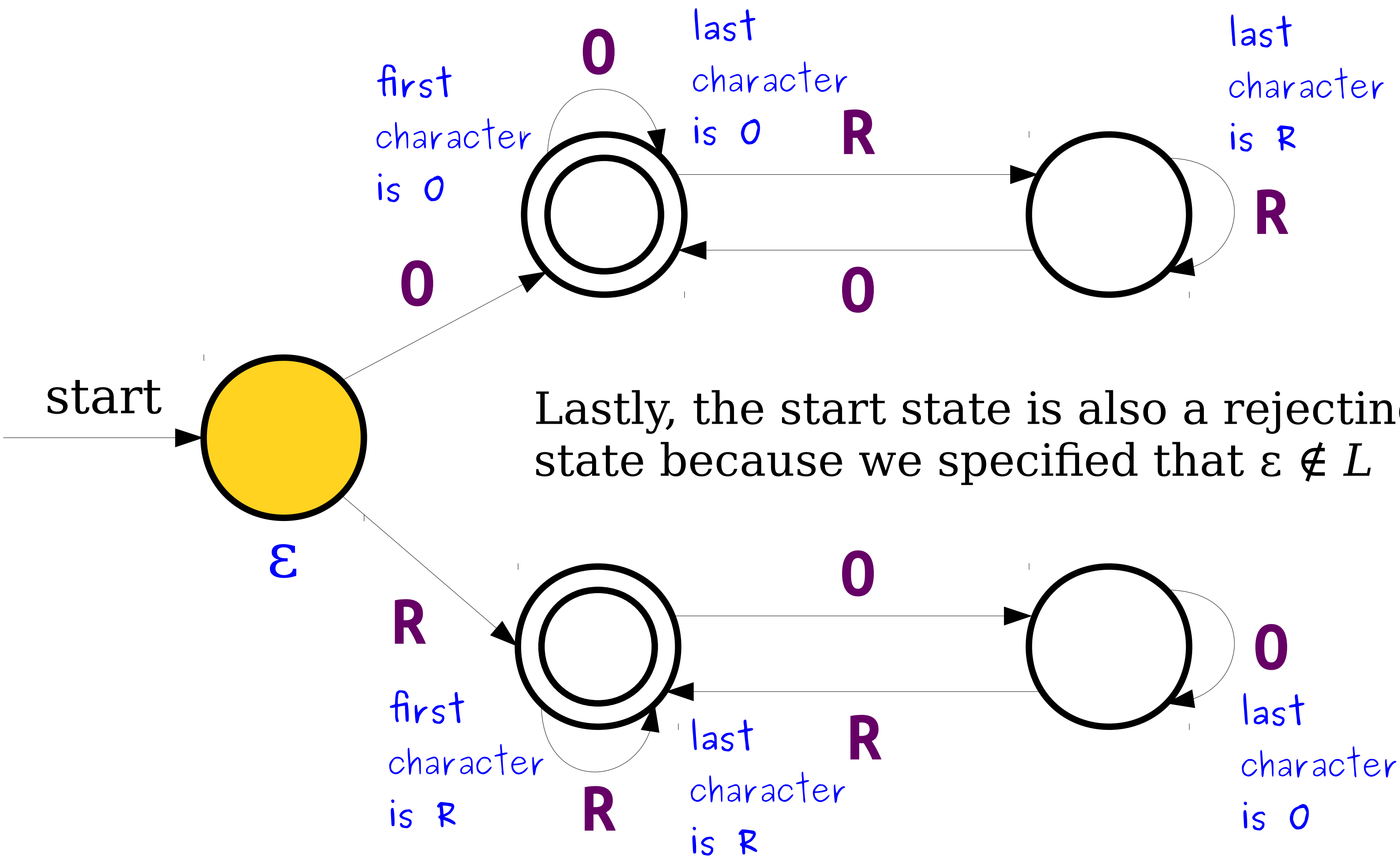
Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



Oreo Sandwiches

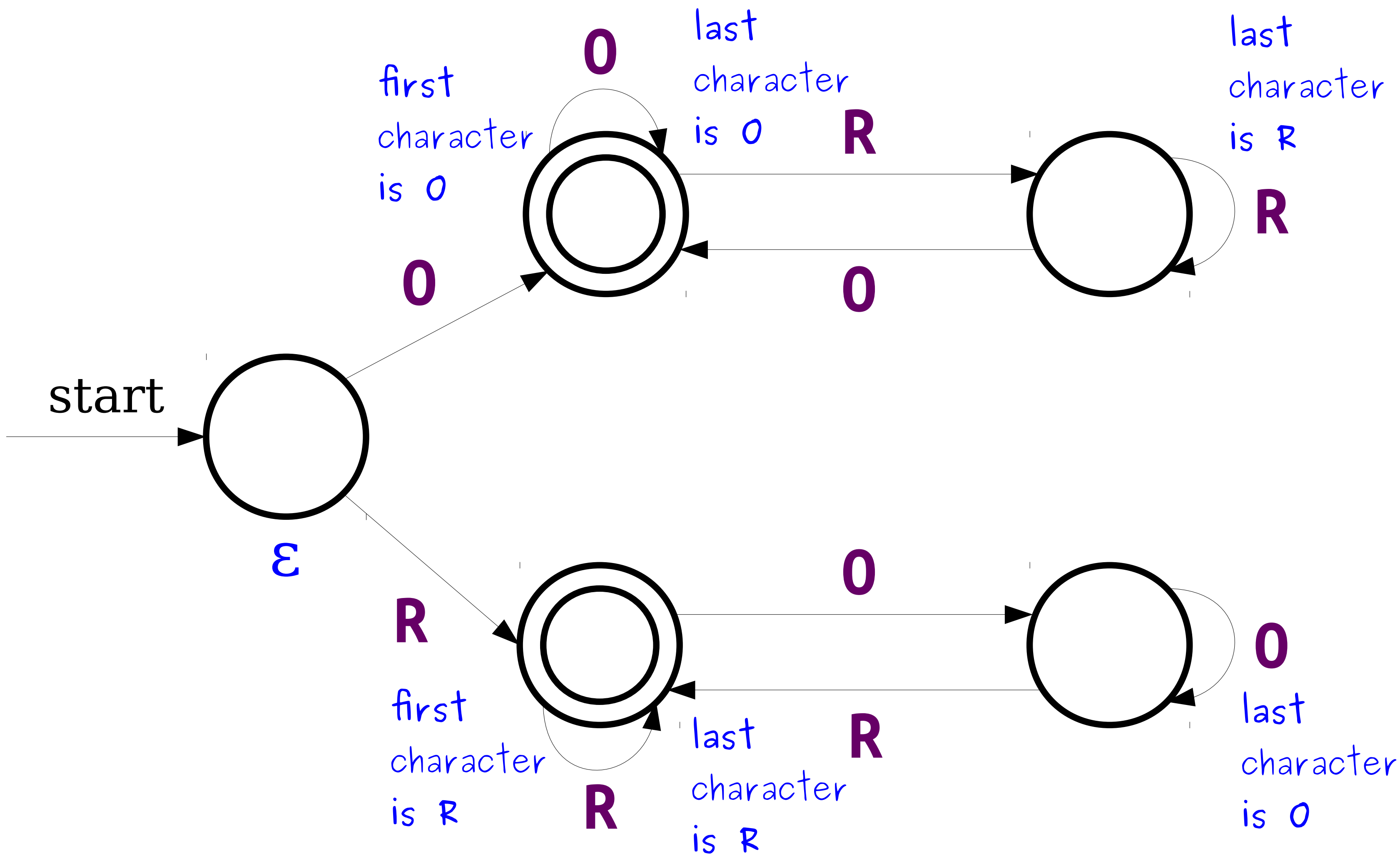
$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



Lastly, the start state is also a rejecting state because we specified that $\varepsilon \notin L$

Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



Oreo Sandwiches

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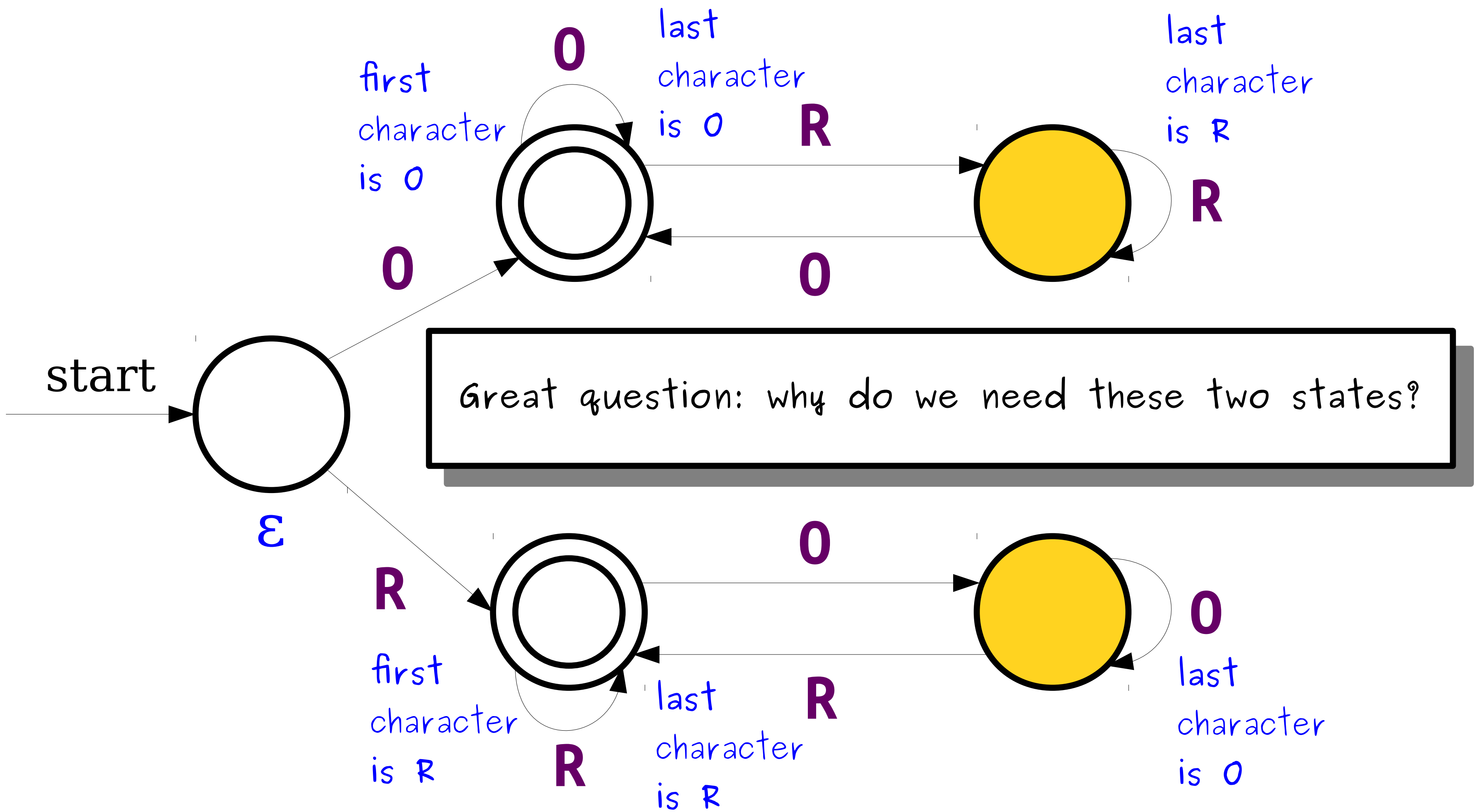
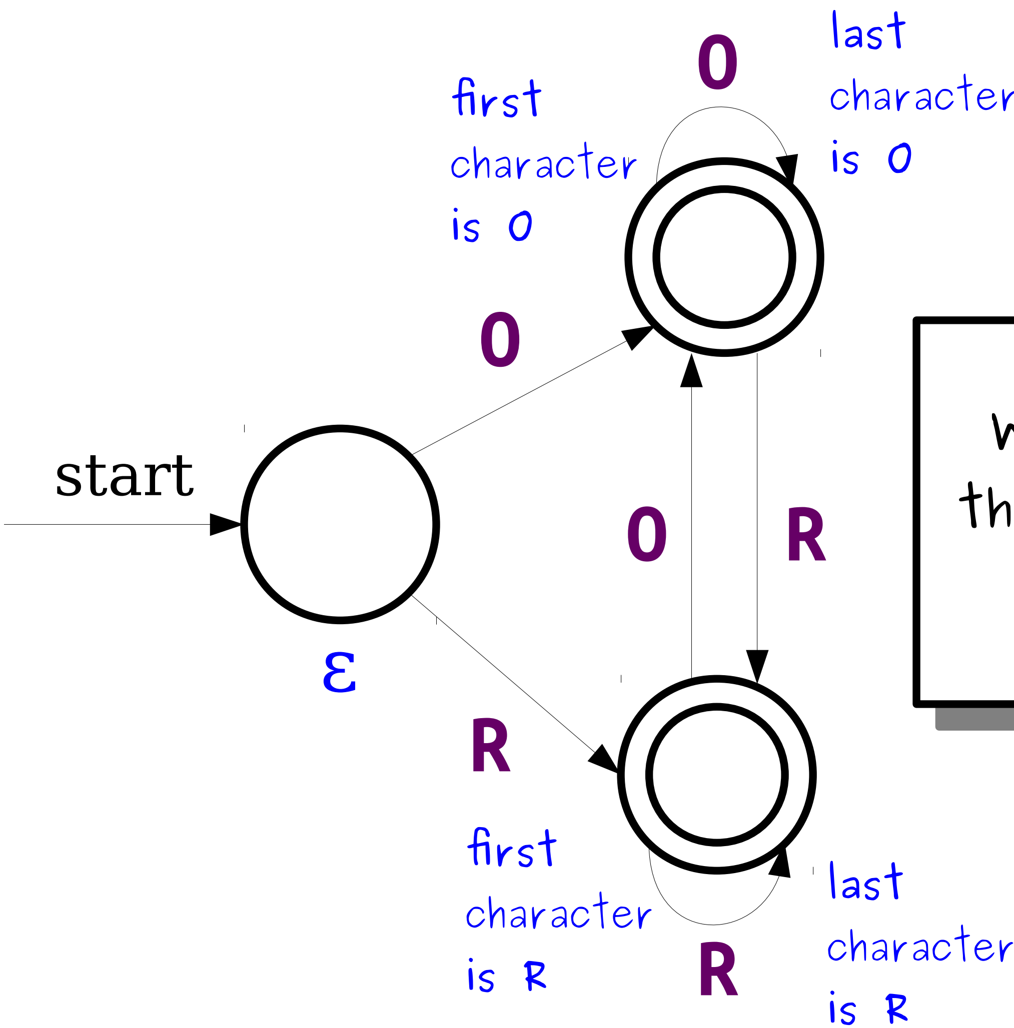


Oreo Sandwiches

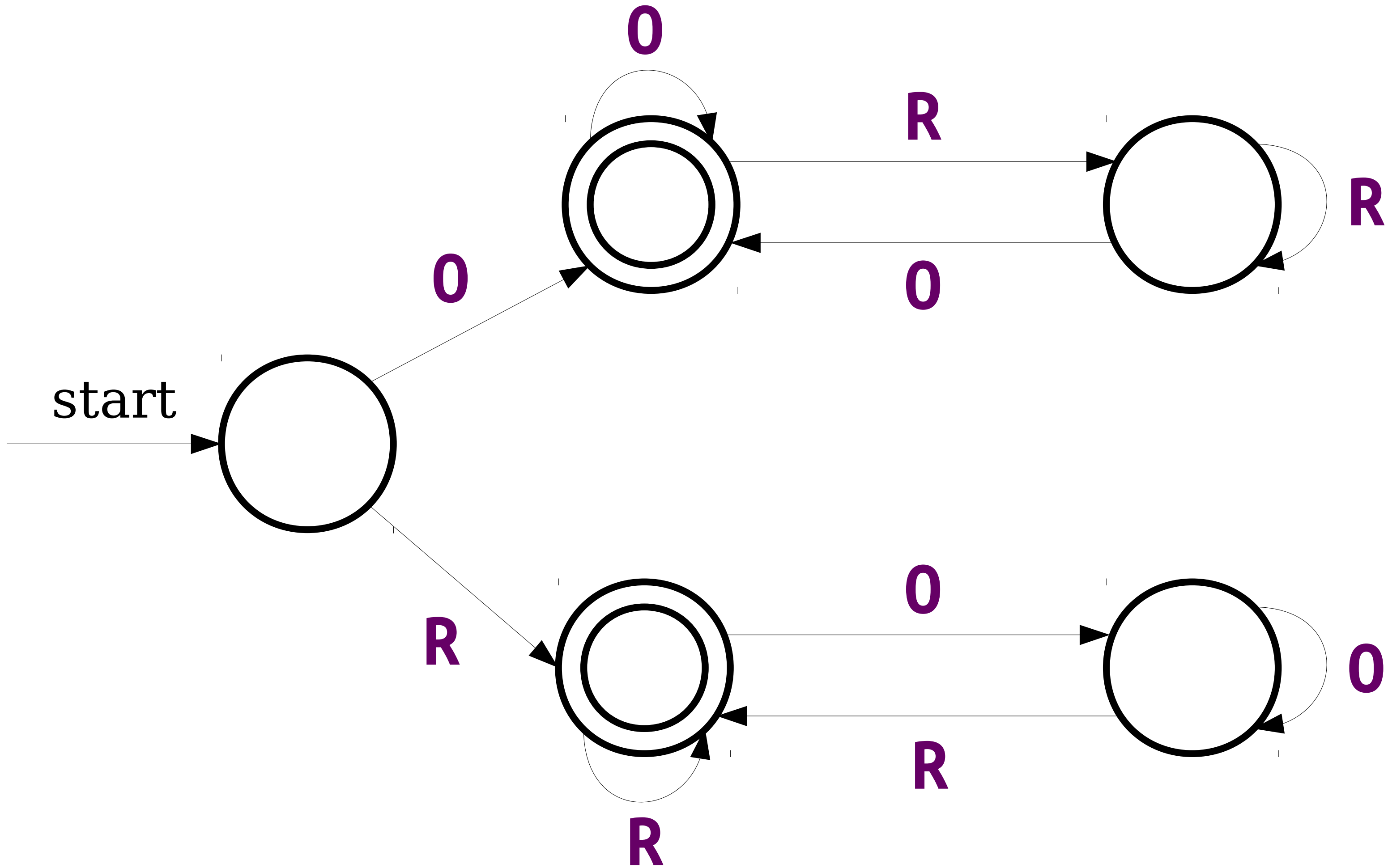
$L = \{\ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



Why can't we have a DFA that looks like this for this language?

Oreo Sandwiches

$L = \{\ w \in \Sigma^* \mid w \neq \varepsilon$ and the first and last character of w are the same $\}$



More Oreo Sandwiches

- Let $\Sigma = \{$ **O**, **R** $\}$

Design a regex for the language

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the characters of w alternate between **O** and **R** $\}$

More Oreo Sandwiches

- Let $\Sigma = \{$ **O**, **R** $\}$

Design a regex for the language

$$L = \{\ w \in \Sigma^* \mid w \neq \varepsilon \text{ and the characters of } w \text{ alternate between } \mathtt{O} \text{ and } \mathtt{R}\ \}$$

ORO $\in L$	**OOR** $\notin L$
RORORor	

Designing Regexes

Write out some sample strings in the language and look for patterns:

- Can I separate out the strings into two (or more) categories?
  - ***Union*** – find the pattern for each category, then union together
- Can I break this problem down into solving some smaller subproblems?
  - ***Concatenation*** - find the pattern for each piece/subproblem, then concatenate together
- Is there some sort of repeating structure?
  - ***Kleene star*** – find smallest repeating unit, then star that pattern

More Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the characters of w alternate between **O** and **R** $\}$

O

R

OR

RO

ORO

ROR

OROR

RORO

ORORO

ROROR

...

Here's one way we could design this regex

More Oreo Sandwiches

$L = \{\ w \in \Sigma^* \mid w \neq \varepsilon$ and the characters of w alternate between **O** and **R** $\}$

O

R

OR

RO

ORO

ROR

OROR

RORO

ORORO

ROROR

...

Can I separate out the strings into two (or more) categories?

- ***Union*** – find the pattern for each category, then union together

More Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the characters of w alternate between **O** and **R** $\}$

Starts with **O**	Starts with **R**
O	**R**
OR	**RO**
ORO	**ROR**
OROR	**RORO**
ORORO	**ROROR**
...	**...**

Can I separate out the strings into two (or more) categories?

- ***Union*** – find the pattern for each category, then union together

More Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the characters of w alternate between **O** and **R** $\}$

Starts with **O**	Starts with **R**
O	**R**
OR	**RO**
ORO	**ROR**
OROR	**RORO**
ORORO	**ROROR**
…	**…**

More Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the characters of w alternate between **O** and **R** $\}$

Starts with **O**	Starts with **R**
O	**R**
OR	**RO**
ORO	**ROR**
OROR	**RORO**
ORORO	**ROROR**
...	...

Can I break this problem down into solving some smaller subproblems?

- ***Concatenation*** - find the pattern for each piece/subproblem, then concatenate together

More Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the characters of w alternate between **O** and **R** $\}$

Starts with **O**	Starts with **R**
O	**R**
OR	**RO**
ORO	**ROR**
OROR	**RORO**
ORORO	**ROROR**
...	**...**

Can I break this problem down into solving some smaller subproblems?

- ***Concatenation*** - find the pattern for each piece/subproblem, then concatenate together

O(sequence of **RO**s)(possibly another **R**)

More Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the characters of w alternate between **O** and **R** $\}$

Starts with **O**	Starts with **R**
O	**R**
OR	**RO**
ORO	**ROR**
OROR	**RORO**
ORORO	**ROROR**
...	**...**

O(**RO**)***R**?

Is there some sort of repeating structure?

- ***Kleene star*** – find smallest repeating unit, then star that pattern

More Oreo Sandwiches

$L = \{ w \in \Sigma^* \mid w \neq \varepsilon$ and the characters of w alternate between **O** and **R** $\}$

Starts with **O**	Starts with **R**
O	**R**
OR	**RO**
ORO	**ROR**
OROR	**RORO**
ORORO	**ROROR**
...	**...**

O(**RO**)***R**? ∪ **R**(**OR**)***O**?

Next Time

- ***Applications of Regular Languages***
  - Answering "so what?"
- ***Intuiting Regular Languages***
  - What makes a language regular?
- ***The Myhill-Nerode Theorem***
  - The limits of regular languages.